WAVE PROBLEMS WITH SPACE-DEPENDENT BOUNDARY CONDITIONS

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The usual linearised condition for waves of frequency $\omega/2\pi$ in water of depth H is

$$K\phi - \partial\phi/\partial z = 0 \qquad (1)$$

where ϕ is the time-independent velocity potential, z is measured vertically upwards, and $K = \omega^2/g$.

There exist situations in which it is desirable to seek harmonic functions ϕ which satisfy (1) when K is replaced by a function of the horizontal space co-ordinates. For example when

$$K = K(x) = K_1 + (K_2 - K_1)H(x) \qquad (2)$$

where H is the Heaviside step function, the equations can describe the propagation of waves from $x = -\infty$ encountering a region $(x > 0)$ of small floating surface particles. Here $K_1 = \omega^2/g$ as before whilst $K_2 = \frac{\omega^2}{g}(1 - sK_1h)^{-1}$ where s is the specific gravity of the particles, and h their thickness. This problem was solved by Peters [1] in deep water and extended by Weitz and Keller [2] to finite water depth and oblique incidence, using the Wiener-Hopf technique. If we wished to allow for a smoothly varying change in surface density of the surface particles we might choose

$$K(x) = (K_1 + K_2 \exp(x/a)/(1 + \exp(x/a)) \qquad (3)$$

with $a > 0$, which approaches (2) as $z \to 0$.

It turns out that with this form for K(x) the problem can still be solved explicitly and results for $|R|$ the magnitude of the reflection coefficient can be obtained with little labour. In fact the boundary-value problem is identical to that solved by Roseau [3] in considering the reflection of waves over a bottom topography of a special type. After conformally mapping the fluid region he obtained the present problem in the transformed plane. He solved the problem in a long paper by means of Fourier-type integrals, reducing the problem to the solution of a functional-difference equation.

A simplified description of the technique as it applies to the present problem and its extension to oblique waves is given in a recent paper by Evans [4] where further applications of the technique to other physical problems are suggested.

In the context of water-wave problems the method is appropriate whenever the wavenumber of free waves at one infinity changes smoothly for whatever reason to a different constant value at the other infinity.

In the present paper we consider a further extension of the method to the problem of waves propagating through fast ice modelled as a thin elastic plate covering the water surface (Squire [8]). The ice thickness is assumed to vary smoothly from one constant value to another so that the resulting wave-number varies also with distance along the free-surface. To simplify matters, the shallow-water equations will be used although it is anticipated that the technique will work for the full linear theory also. The advantage of the shallow-water approximation, despite it being restricted to waves which are much longer than the water depth, is that only ordinary differential equations are involved although the coefficients are space-dependent.

With x,y as horizontal co-ordinates, z vertically upwards, the velocity potential $\Phi(x,y,z,t)$ satisfies

$$\Phi_{tt} + g\eta_t = -\frac{Eh^3(x)}{12\rho(1-\nu^2)}\nabla^4\eta_t - \frac{\rho_i h(x)}{\rho}\eta_{tt}$$

where

$$\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$$

and $\eta(x,y,t)$ is the surface elevation. Here E is Young's modulus, ν is Poisson's ratio, $\rho_i,(\rho)$ the density of the ice, (water) and $h = h(x)$ the ice thickness assumed to vary in the x-direction only.

Now on shallow-water theory

$$\eta_t = (h - H)\nabla^2\Phi$$

where H is the water depth. If we assume $h(x)/H << 1$ and look for a solution $\Phi = \text{Re}\ \phi(x,z)\exp(-i\omega t)$, we obtain

$$M(x)\frac{d^6\phi}{dx^6} + \frac{d^2\phi}{dx^2} + k_o^2\phi = 0 \qquad (4)$$

where $M(x) = Eh^3(x)/12\rho g(1 - \nu^2)$

and $k_o^2 = \omega^2/gH.$

We now assume

$$M(x) = (M'\exp(x/a) + M)/(\exp(x/a) + 1). \qquad (5)$$

For example if $M = 0$ this $M(x)$ describes an ice cover whose thickness varies smoothly from zero at $x = -\infty$ to a constant value determined by $M' = Eh^3(\infty)/12\rho g(1 - \nu^2)$ at $x = +\infty$. With $M' \neq 0$ the thickness varies smoothly over a length scale $O(a)$ from one constant value at $x = -\infty$ to a different value at $x = +\infty$. The limit problem $M = 0$, $a \to 0$ corresponding to a sudden change in ice thickness from zero to a constant value at $x = 0$ which is discussed in Stoker [5] and was extended to oblique incidence by Evans and Davies [6].

Equations (4) and (5) can be written

$$e^x(M'\phi_x^{(6)} + \phi_{xx} + k_o^2\phi) + M(\phi_x^{(6)} + \phi_{xx} + k_o^2\phi) = 0$$

where $a = 1$ without loss of generality.

Now we can show that a contour integral solution of the form

$$\phi(x) = \frac{1}{2\pi i}\int_C \frac{e^{ikx}f(k)dk}{M'k^6 + k^2 - k_o^2}$$

exists provided

$$\text{a)}\quad \frac{f(k+i)}{f(k)} = -\frac{Mk^6 + k^2 - k_o^2}{M'k^6 + k^2 - k_o^2} \tag{6}$$

and b) the contour C is chosen suitably.

A similar approach was used by Levine [7] for a second-order differential equation with variable coefficients. Both of these conditions can be satisfied and $f(k)$ determined such that ϕ behaves in the required way at $\pm\infty$. In particular it is found that

$$|R| = \sinh\pi(k_1 - k_1')/\sinh\pi(k_1 + k_1')$$

where k_1, k_1' are the real positive roots of

$$Mk^6 + k^2 - k_o^2 = 0, \quad M'k^6 + k^2 - k_o^2 = 0$$

respectively.

Returning to the full linear problem, the functional-difference equation which needs to be solved is

$$\frac{F(k+i)}{F(k)} = \frac{K - k(1 + Mk^4)\tanh kH}{K - k(1 + M'k^4)\tanh kH} \tag{7}$$

which can be treated using either Fourier transforms or infinite product decompositions. It is anticipated that in this case

$$|R| = \sinh\pi(k_o - k_o')/\sinh\pi(k_o + k_o')$$

where k_o, k_o' are the real positive roots of the numerator and denominator of (7) respectively.

References

1. A.S. Peters (1950) The effect of a floating mat on water waves. Comm. Pure Appl. Math. 3, 319-54.

2. M. Weitz and J.B. Keller (1950) Reflection of water waves from floating ice in water of finite depth. Comm. Pure Appl. Math. 3, 305-18.

3. M. Roseau (1976) Asymptotic Wave Theory. North Holland, 1976.

4. D.V. Evans The solution of a class of boundary-value problems with smoothly varying boundary conditions. Quart. J. Mech. Appl. Math. (In press).

5. J.J. Stoker Water Waves. Interscience 1957.

6. D.V. Evans and T.V. Davies (1968) Davidson Laboratory Report No. 1313. Stevens Institute of Technology.

7. H. Levine (1982) On a mixed boundary value problem of diffusion type. Applied Sci. Res. 39, 261-76.

8. V. Squire (1984) A theoretical, laboratory, and field study of ice-coupled waves, J.G.R., Vol. 89, No. C5, 8069-8079.

Discussion

Tuck: Are these calculations for constant depth?

Marshall: Yes, they are for constant thickness at either infinity or constant depth of water.

Troesch: Did you allow for any energy dissipation in your ice model? How important is it?

Marshall: No, there is no allowance for energy loss. I believe V. Squires investigated mechanical dissipation. The effect was not too great.

Mehlum: There are reports from seamen navigating in the Arctic that the largest wave amplitude waves occur some distance (a few hundred meters to one kilometer) into the free-ocean side from the edge of the pack ice. Do you see this feature in your model?

Marshall: No, we have not investigated amplitudes.

Evans: Perhaps I could make a comment on the applicability of the method which is based on Roseau's work. It appears it can be extended to any problem in which the wavenumbers change smoothly from one infinity to the other in a particular manner. For example, the problem of acoustic waves travelling down a wave guide bounded by rigid walls into a region where an abrupt change to an impedance condition occurs can be solved using the Wiener-Hopf technique. We can use the present method to solve for a smooth change from the rigid to impedance condition and get the result for $|R|$ fairly quickly. The result goes over to the result for an abrupt change in the appropriate limit.

Kleinman: Your choice of smoothly varying wavenumber is interesting. However, there are many ways a function can smoothly vary from k_1 to k_2. How important is the choice?

Evans: It is crucial. It has to be the ratio of exponentials to obtain a functional difference equation.

Tuck: I compliment the authors on understanding Roseau's paper. Now there may be three times as many people in the world who understand that paper. I am not one of them!

Yeung: In computing reflection and transmission coefficients of waves propagating over a step on the bottom, I found that the discontinuity of the bottom slope causes these coefficients to oscillate as functions of frequency.

Agnon: It would be interesting to look at group velocity.

Evans: The group velocity techniques have not been investigated in detail so far.