APPLICATION OF THE LOCALIZED FINITE ELEMENT METHOD TO THE 2-D NEUMANN-KELVIN PROBLEM

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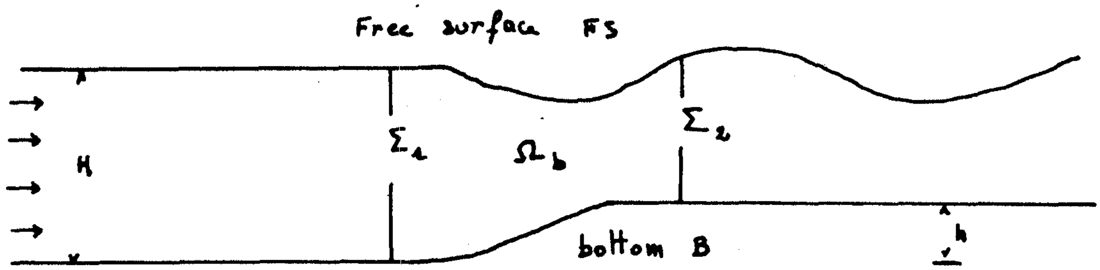
The aim of this work is to solve the 2-D nonlinear wawe resistance problem. One possible way is to use the localized finite element method (L.F.E.M.). In this way we can solve the nonlinear equations near the bottom irregularities and the linearized equations elsewhere. The L.F.E.M. has been originally developped by K.S BAI [1] for numerical solutions of the 2-D Neumann-Kelvin problem.

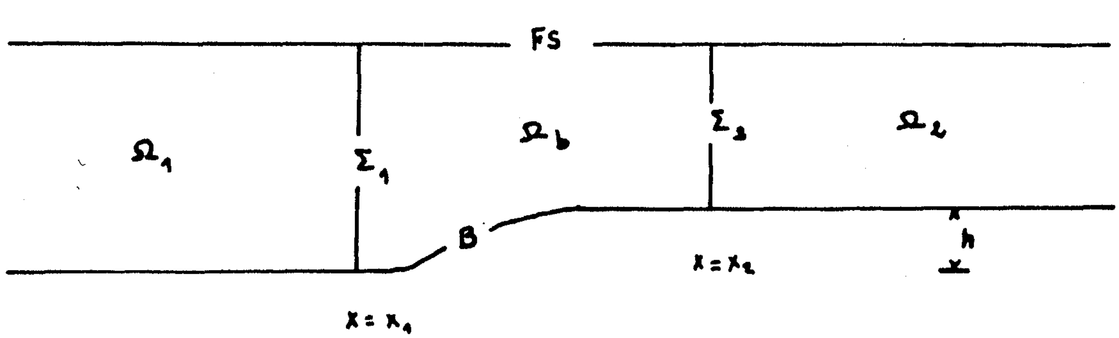


We consider an incompressible perfect fluid flow irrotational at infinity on an irregular bottom. The bottom irregularities have a compact support. The problem can be formulated in stream function or in potential function.

I - THE STREAM FUNCTION FORMULATION

I.1 The linear case a result of existence and uniqueness

The free surface and bottom equations are respectively $y=1$ and $y=f(x)$.



The dimensionless equations for the perturbation of the stream function ψ and for η the perturbation of the linearized free surface are given by the problem (P) : Find ψ fulfilling :

$$\Delta\psi = 0 \text{ in } \Omega,$$
$$\partial_n\psi = F^{-2}\ \psi \text{ on } Fs,$$
$$\psi+f = 0 \text{ on } B,$$
$$\psi \text{ and } |\nabla\psi| \text{ are bounded and vanish at infinite upstream.}$$

F denotes the Froude number.

Theorem 1.1

For $F < 1$, if f is piecewise continously differentiable, if $f' > 0$, and if $F^{-2}(1-H/h) \neq 1$, then there exist a unique $\psi \in H^1_{\ell oc}(\Omega)$ solution of (P). This result is proved in Pousin [2].□

Formulated in this way we cannot solve numerically this problem because the fluid domain is unbounded. Therefore we introduce a problem posed in a bounded domain. Let us suppose that we know ψ_b a solution of the resticted problem (P) to Ω_b and ψ_i solutions of the restricted problem (P) to Ω_i, $i = 1,2$. If these functions connect each other harmonically then $\psi = \psi_b$ in Ω_b, $\psi=\psi_i$ in Ω_i will be a solution of (P). For this it is sufficient to impose the connection in value and in normal derivative. Now we are able to introduce the matching operator

$T_i\ H^{1/2}(\Sigma_i) \to H^{-1/2}(\Sigma_i)$ which associates $\partial_n u_i$ to $\phi \in H^{1/2}(\Sigma_i)$ where u_i is solution of :

$$\Delta u_i = 0 \text{ in } \Omega_i,$$
$$\partial_n u_i = F^{-2}\ u_i \quad \text{on } F s_i,$$
$$u_i = 0 \text{ on } B_i$$
$$u_i = \phi \text{ on } \Sigma_i,$$
$$u_i \text{ and } |\nabla u_i| \text{ are bounded and vanish at infinity upstream.}$$

Looking for u_i in form of a separate variables function, we have the following result :

Proposition 1.1.

The functions depending on y in the separate variables function u_i make up a basis of $H^1(\Sigma_i)$ and $H^{1/2}(\Sigma_i)$ and are orthogonal in $L^2(\Sigma_i)$. Moreover if $\phi \in H^2(\Sigma_i)$ and $\phi(h_i) = 0$, then $T_i\phi$ can be represented as a series.

Before introducing the problem posed in a bounded domain (Pb) we need to extend the Dirichlet condition on the bottom B_2 by a function g which satisfies : support of $g \subset \Omega_b \cup \Omega_2$, support of $\Delta g \subset \Omega_b$, g fulfils the free surface condition. Then we shift ψ and denote again by ψ, $\psi - g$. □

I.2 The bounded domain (Pb)

Find $(\psi,\alpha) \in H^1(\Omega_b) \times \mathbb{R}$ solution of :

$\Delta\psi = 0$ in Ω_b,

$\partial_n\psi = F^{-2}\psi$ on FS_b,

$\partial_n\psi = - T_i\psi$ on Σ_i $\quad i = 1,2$

$\psi + f = 0$ on B_b,

$(\psi, \Psi_1^0)_{L^2(\Sigma_1)} = 0$, $(\partial_n\psi, \Psi_2^0)_{L^2(\Sigma_2)} = \alpha$, where

$\Psi_1^0 = \text{sh}\, \omega_1^0\, y$, $\text{th}\, \omega_1^0 = \omega_1^0 F^2$, $\Psi_2^0 = \text{sh}\, \omega_2^0\, (y-h)$, $\text{th}(1-h)\omega_2^0 = \omega_2^0 F^2$

Theorem 1.2

The problems (P) and (Pb) are equivalent.

Now we can consider the nonlinear problem where the fully Bernouilli equation is locally treated. For a description of the equations and an algorithm alloving to compute numerical solutions look in [3]. □

II - THE POTENTIAL FUNCTION FORMULATION

II.1 The linearized equations

The dimensionless problem (Q) for ϕ the perturbation of the potential function and for η the perturbation of the linearized free surface is :

Find $\phi \in \{v \in H^1_{loc}(\Omega),\ \partial_x v \in L^2_{loc}(SL)\}$ satisfaying :

$\Delta\phi = 0$ in Ω,

$\partial_n\phi = - F^2\, \partial_x^2\phi$ on FS,

$\partial_n\phi = -(\vec{n}/\vec{x})$ on B,

ϕ and $|\nabla\phi|$ are bounded and vanish at infinite upstream.

$\vec{n}$ is the outward normal. Then we can deduce the free surface elevation a posteriori by $\eta + F^2\, \partial_x\phi = 0$ on FS. □

As for the stream function formulation we need to introduce a problem posed in a bounded domain. Let us consider the following operator T_i.

$T_i : H^{1/2}(\Sigma_i) \to H^{-1/2}(\Sigma_i)$ which associates the normal derivative of u_i to ϕ where u_i is solution of :

$$\Delta u_i = 0 \text{ in } \Omega_i,$$
$$\partial_n u_i = - F^2 \partial_x^2 \phi \text{ on } FS_i,$$
$$\partial_n u_i = 0 \text{ on } B_i,$$
$$u_i = \phi \text{ on } \Sigma_i,$$

u_i and $|\nabla u_i|$ are bounded and vanish at infinite upstream.

We look for a representation of T_i in form of a series. To do this we put $u_i = v(y)\, f(x)$ then we get the following result :

Proposition 2.1.

The functions $v(y)$ make up a basis of $H^1(\Sigma_i)$. Morever for each $w \in H^1(\Sigma_i)$ we can represent it as a series :

$$W(y) = \sum_{k=o}^{+\infty} [b^i(W,\phi_i^k)/b^i(\phi_i^k/\phi_i^k)]\ \phi_i^k(y) + \alpha[b^i(W,\phi_i^{oo})/b^i(\phi_i^{oo},\ \phi_i^{oo})]\ \phi_i^{oo}(y),$$ where ϕ_i^k

are the basis functions and b^i is the bilinear form defined by

$$b^i(u,v) = \int_{h_i}^{1} v(y)u(y) - F^2\ u(1)v(1) \qquad h_i = (i-1)h, i = 1,2.$$

The matching operator

We introduce the matching operator $(T_i + \beta_i\ Id)$, $\beta_i \in R$.

Lemme 2.1

If $u_i \in H^2(\Sigma_i)$ we have :

$$(T_i + \beta_i\ Id)u_i = \sum_{k=1-(i-1)}^{+\infty} (\omega_i^k + \beta_i)\ Q_i^k(u_i)\ \phi_i^k(y) +$$
$$(i-1)\alpha(\omega_i^{oo} + \beta_i)\ Q_i^{oo}\ (u_i)\ \phi_i^{oo}(y)$$

where $Q_i^k(u_i)$ are the components of u_i on the basis.

Remark

If β_1 is different from 0, as $b^1(\phi_1^k,\ \phi_1^J) = \delta_{kJ}$, then u_1 cannot be constant.□

Now we consider the problem posed in a bounded domain (Qb) :

Find $(\phi,\alpha) \in \{ V \in H^1(\Omega_b),\ \partial_x v \in L^2(FS_b)\} \times R$ solution of :

$\Delta\phi = 0$ in Ω_b,

$\partial_n\phi = - F^2\, \partial_x^2\phi$ on FS_b,

$\partial_n\phi - \beta\phi = -(T_1 + \beta)\,\phi$ on Σ_1,

$\partial_n\phi = - T_2\,\phi$ on Σ_2,

$\partial_n\phi = - (\vec{n}/\vec{x})$ on B_b.

$b^1(\phi, \phi_1^{oo}) = 0$, $b^2(\partial_n\phi, \phi_2^{oo}) = \alpha$, where $\phi_1^{oo}(y) = ch\omega_1^{oo}y$, $\phi_2^{oo}(y) = ch\omega_2^{oo}(y-h)$,

$th\ \omega_i^{oo}(1-h_i) = F^2\, \omega_i^{oo}$

<u>Remark</u>

In the variational formulation if we use the bilinear form $b^i(.,.)$ to represent the coupling terms, the pinpoint terms $F^2\, \partial_x\phi(x,\eta(x))$ Coming from the integration by parts on the free surface FS_b disappear.□

III - THE NONLINEAR EQUATIONS

The equations are the Bernoulli and kinematic equations on the free surface FS_b. We use an algorithm based on a fixed point method of the geometry of Ω_b and a connection with the solution of the linearized equations outside of Ω_b.

For Ω_b^n, η^n given we compute $(\phi^{n+1}, \alpha^{n+1})$ by :

(*) $\Delta\phi^{n+1} = 0$ in Ω_b,

$\partial_n\phi^{n+1} = - F^2\, \partial_S^2\, \phi^{n+1} + k(\eta^n, \eta'^n)$ on FS_b,

$\partial_n\phi^{n+1} = - (\vec{n}/\vec{x})$ on B_b,

$\partial_n\phi^{n+1} - \beta\phi^{n+1} = -(T_1 + \beta)\,\phi^{n+1}$ on Σ_1

$\partial_n\phi^{n+1} = - T_2\,\phi^{n+1}$ on Σ_2

$b^1(\phi^{n+1}, \psi_1^{oo}) = 0$, $b^2(\partial_n\phi^{n+1}, \psi_2^{oo}) = \alpha^{n+1}$

then we compute η^{n+1} by :

$$\eta^{n+1}(z) = \int_{FS_b^n \cap \{x_1 \leqslant z\}} \partial_n\, \phi^{n+1}\, ds + \eta^{n+1}(x_1), \text{ where } \eta^{n+1}(x_1) \text{ is}$$

evaluted with the linearized model.

If Ω_b^{n+1} is different from Ω_b^n Go to (*) .

Some Comments.

On the free surface we use a mixed condition of the Bernoulli and kinematic equations because without a condition such as the linearized free surface condition the L.F.E.M. doesnot imply a propagative solution in Ω_2.

As we use the real free surface at each step, even if we use $b^i(.,.)$ to represent the coupling terms, the pinpoint terms coming from the integration by parts on FS_b dont disappear. Morever the matching operator doesnot connect the tangent derivative at the free surface of the inner and outer solution.

Therefore this algorithm doesnot converge in the fluvial case and converges in the torrential case. In fact in the torrential case the tangential derivative at the free surface is not really different from ∂_x which is the normal derivative on Σ_i. In oder to be able to obtain numerical solutions in the fluvial case we have to use an operator T_i which associates the derivative in the direction of the tangent at the free surface of u_i and not the normal derivative. If we do this and if we represent the coupling terms with $b^i(.,.)$ the pinpoint terms will desappear. □

[1] K.S. BAI : *"A localized finite element method for steady two dimensional free surface flow problem"*. Proceeding of the first international conference on numerical ship hydrodynamics, Gaithersburg, Maryland 1975.

[2] J. POUSIN : *"Un résultat d'existence et d'unicité pour le problème de Neumann-Kelvin"*. Comptes rendus à l'Académie des Sciences, Paris, t. 301 série I n° 20 - 1985.

[3] J. CAHOUET, M. LENOIR : *"Résolution numérique du problème bidimensionnel de la résistance de vagues non linéaire"*. Compte Rendu à l'Académie des Sciences, Paris série II t. 297 - 1983.

Discussion

Kleinman: What geometric restrictions were there on the bottom contour?

Pousin: The restriction is that the slope is positive. For a general bump we have to study the spectrum of the wave operator and we have to prove that there are no eigenvalues for a Froude number different from unity.

Kleinman: At various stages you cited a need for a basis, did you formally show the existence of an orthonormal basis or did you explicitly produce this basis?

Pousin: We have shown the existence of an orthonormal basis in the stream-function formulation. In the potential function formulation we have shown the existence of a basis which is not orthogonal for the usual scalar product of L^2 but which is orthogonal for the bilinear form b^i.

Yeung: The orthogonality is not very clear in J. Bai's paper. It is more than just a dot product, it also has a free surface contribution.

Pousin: The basis is not orthogonal for the potential function formulation but it does not matter if we use the bilinear form b^i in the variational formulation of the problem.

Tuck: Are you aware of Forbes' work published in the Journal of Fluid Mechanics in 1982?

Pousin: Yes, I am familiar with his work.

Mei: Prof. Yeung's comment reminds me of work I did with Chen (1976, J. Num Methods of Engineering) where we have already used that eigenfunction set.

Yeung: But that was with a complex formulation.

Mei: The basis is made up of real eigenfunctions.

T. Wu: The point is that that was for a linear problem, right?

Yue: For the linear problem Mei and Chen (1976) showed that this steady problem can be reduced to two equivalent diffraction problems: a scattering problem and a fictitious radiation problem.

Pousin: I think the formulation proposed by Mei and Chen is not convenient to treat the nonlinear problem. Therefore, we needed to study the localized finite element method.