The Nonlinear Evolution of Viscous Fingers - A Water-wave Problem in Disguise

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It is well-known that single-phase flow through porous media is governed by Darcy's law, i.e. fluid velocity is proportional to pressure gradient. Mass conservation then leads to a potential flow problem. Of interest to oil production is the case where one liquid is used to push another out of a porous oil-bearing formation or "reservoir." Realistically this is a two-phase flow problem where the driving and driven fluids are intimately mixed on the pore level. However, a useful idealization is to assume that the fluids are immiscible and that the microscopic displacement process is complete. This leads to two potential-flow domains separated by a moving boundary, the time evolution of which needs to be determined as part of the solution. While formally a viscous creeping flow, it is mathematically analogous to inviscid free-surface and interfacial wave problems and also to flows resulting from the Rayleigh-Taylor instability.

Of greatest interest is the case where the pusher fluid is the less viscous of the two. The fluid interface is then unstable to small perturbations. Often called the Saffman-Taylor instability[1], the resulting flow exhibits viscous "fingers" where the protrusions of the driving liquid grow without limit. When viewed on a macroscopic scale, much of the driven fluid is bypassed by the moving front, leading to reduced values of "sweep efficiency."

A Hele-Shaw cell, consisting of two glass plates separated by a small uniform distance, is an experimental device used to visualize these unstable displacements. In a rectilinear channel a single dominant finger forms and, at sufficiently low flow rates, may be observed to propagate without change of form. Techniques developed for the calculation of steady-state finite-amplitude Stokes waves have recently been implemented to find the shape of these fingers. The standard idealization neglects the wetting layer of the driven fluid left behind on the walls of the device and averages the flow properties in the direction perpendicular to the plates. The resulting potential flow problem is two-dimensional. The viscous contribution to the normal stress balance on the interface is neglected; thus the pressure jump across the interface is given as the product of the interfacial tension and the curvature. Because the differential system is second-order, the tangential stress balance and the side-wall no-slip conditions must be abandoned. For a given initial interface shape, cell geometry and viscosity ratio, the time-dependent problem utilizes the standard kinematic boundary condition to advance the moving boundary. Only a single dimensionless parameter governs the subsequent motion, the ratio of viscous to surface tension forces, i.e. a capillary number. Within the porous-medium flow community, this system, including the above boundary conditions, is usually referred to as the Hele-Shaw equations.

Several flow geometries have been considered[2,3,4]. As an example, we will state the problem that has been treated most extensively, that of fingering in a rectilinear channel when the pusher fluid is taken to be inviscid. Darcy's law takes the form

$$\bar{v} = -M\nabla p \qquad (1)$$

$$M = b^2/(12\mu)$$

where b is the plate spacing and μ is the liquid viscosity. Assuming the liquid to be incompressible, the continuity equation is

$$\nabla\cdot\bar{v} = 0 \cdot \qquad (2)$$

Combining the two equations, we obtain Laplace's equation for the pressure in the liquid:

$$\nabla^2 p = 0 \quad \cdot \qquad (3)$$

On the interface we have

$$p(x,y) = \sigma\,\frac{d\theta}{ds}\,, \qquad (x,y)\ \epsilon\ \partial R \cdot \qquad (4a)$$

where s is the arc length and θ is surface inclination. On the moving interface ∂R the kinematic boundary condition, that a particle on this boundary remain on the boundary for all time, must also be satisfied. Restated, this condition is that the normal component of fluid velocity, with cartesian components (u,v), of a particle occupying a point on the surface, is equal to the normal component of the surface velocity (x_t, y_t) at that point. Thus

$$(u,v)\cdot\bar{n} = (x_t, y_t)\cdot\bar{n}, \qquad (x,y)\ \epsilon\ \partial R \qquad (4b)$$

where $\bar{n}$ is a normal vector to ∂R and subscripts signify time differentiation. Because Hele-Shaw flow is a subset of creeping motion, inertial contributions to the interface motion are neglected and surface velocity components found by solving (3), at each instant of time, yield the instantaneous surface motion according to (4b).

On the side walls of the cell, the normal component of fluid velocity is zero, i.e.

$$\frac{\partial p}{\partial x} = 0 \,, \qquad x = \pm L/2 \,. \tag{5}$$

At upstream infinity, we assume constant velocity,

$$M\nabla p \to -\frac{Q}{bL}\,\bar{y} \qquad y \to +\infty \,, \tag{6}$$

where Q is the volumetric flow rate into the channel and $\bar{y}$ is a unit vector in the direction of increasing y.

We have solved this system numerically, using a boundary- integral technique to treat the imbedded linear problem. Time integration is done implicitly. Methods involving both distributed sources and distributed vorticity have been developed and each has certain advantages. The scheme is essentially Lagrangian and a key feature involves periodic redistribution of the boundary nodes. For a rectilinear channel, a slightly-perturbed interface evolves into a steady-state propagating finger, in qualitative agreement with experimental results. Previous stability analyses, suggesting that these fingers, if formed, should be unstable, can be understood by recognizing that, while certain small disturbances can be found that will grow initially, ultimately they are "left behind" by the moving finger and will decay.

When the viscosity ratio is unfavorable, it can be shown that the problem without surface tension is ill-posed in the sense that large wavenumber disturbances grow most rapidly. Because the effective surface tension force is very small in laboratory-scale flows (and even smaller when oil-field dimensions are considered), results are critically sensitive to the ambient level of "noise." Thus we have included a second parameter in the model to represent the noise level. In Hele-Shaw cell experiments this noise may arise via imperfections in the device or vibrations in the environment. Since no actual porous medium is totally homogeneous, noise in porous media flows may be identified with local permeability variations. Computations[3] reveal that, for a given value of capillary number, there is a critical noise level. When this critical level is exceeded, the displacement front becomes jagged, with many side branches. Such flows have recently been observed experimentally, in linear Hele-Shaw cells at relatively large capillary number, and are similar in form to results of the simulation. In other geometries, where "flooding" patterns are modelled by a discrete combination of sources and sinks[4], the noise-induced side branching leads to reduced values of breakthrough sweep efficiency, i.e. the portion of the reservoir contacted prior to the pusher fluid reaching a sink. Even when no noise is introduced into the computations, the specific details of the fingering patterns are strongly sensitive to the initial conditions; the large subsequent magnification of disturbances arises because the problem is "almost ill-posed" when the capillary number is large.

Results have recently been extended to finite values of viscosity ratio. As the pusher viscosity is increased, sweep efficiency will also increase. Partly this is due to the fact that fingers fatten with increased pusher viscosity. A more important effect, however, is the large increase in critical noise amplitude. Thus, for a given value of spatial permeability variation, the ratio of viscosities will determine whether or not the displacement front will break up into a tree-like structure.

While the time-dependent solutions of the Hele-Shaw equations, with noise, reproduce the qualitative features observed in experiments, the numerical value of the width of a steadily propagating finger in a straight channel is consistently underestimated. This has been conjectured to be due to the neglect of the additional interface pressure jump associated with the residual layer of displaced liquid left behind on the cell walls. An asymptotic result of Bretherton[5] predicts the magnitude of this effect. By altering the numerical algorithm so that this correction can be applied locally, as an "inner solution", a good match with the available data is obtained. Because this added pressure jump depends on the local value of speed at each point on the front, the imbedded potential problem becomes nonlinear; it can, however, be treated successfully using Newton iteration.

REFERENCES

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Discussion

T. Wu: What are the nonlinear effects?

Schwartz: The problem is nonlinear solely because we trace the interface evolution to large amplitude. When the wetting-layer effect is included, even the embedded boundary-integral problem becomes nonlinear.

Stiassnie: Have you or others tried to apply fractal ideas to your results? Do they get a straight line for the lag (length) to lag (yard-stick) relation?

Schwartz: I do not have enough fine-scale structure in my results to date. In the diffusion limited aggregation (DLA) model, they do get a power law relation for the boundary length. However the DLA model needs to be justified since we are treating a hydrodynamics problem.

Tuck: I should like you to expand upon the role of the second (thickness-wise) component of curvature, which is dominant, and may not be constant.

Schwartz: In the primitive model, the curvature between the plates, while very large, is assumed to be strictly constant. If this curvature is not quite constant, an order-one correction needs to be introduced. Using the nonlinear pressure-velocity relation locally on the moving interface does produce a large change in the profile shape.

Dommermuth: Did you use constant panels? It is my experience that piecewise-constant function values are not effective. Consider a rectangular box of fluid. On the left and right walls, let $\phi_x = 1$, and on the top and bottom let $\phi = x$. So the solution is $\phi = x$. Now use Green's theorem to solve the same problem with constant panels. Near the corners a constant variation in potential cannot approximate a linear gradient. A spurious singularity is introduced (in addition to other weaker singularities) as a result of our approximation. Now consider a standing wave solution ($\phi = e^x \cos x$) near a corner which is not a right angle. Because of our constant panel assumption, the water-particle velocity is zero in three different directions. This too is a singularity. However, panels using linear variations of ϕ make both singularities weaker or may eliminate them altogether.

Schwartz: The source strength was piecewise constant. Because of capillarity, we do not have sharp corners.

Mehlum: Concerning the basic equations, you said the boundary is not sharp. What does that mean?

Schwartz: I consider a boundary to be sharp when the characteristic thickness of the front, i.e., the transition region between the two asymptotic values of residual saturation, is much smaller than the inter-well distance for example. Both sharp and diffuse boundaries can be found in practice. At full scale, there is little data, because of the large expense involved in drilling extra holes, i.e., observation wells.

Yue: You used a random perturbation on the interphase coordinates to simulate an inhomogenous medium. Have you considered using a variable (or indeed stochastic) coefficient in your Darcy's law formulation?

Schwartz: In our model, the interface is quite sharp. The boundary integral formulation would loose most of its power if inhomogeneity strictly within the fluid domain were important. Fortunately, it can be argued that the greatest effect of permeability variation is at the moving interface.

Lea: If you consider the single-phase flow through an artificial packed bed composed of spheres, say, you will find convective instabilities, similar to vortex shedding past a single sphere, being propagated through the porous media. These packets of vorticity will appear as randomly distributed disturbances within the medium. The initiation of these disturbances is Reynolds-number dependent and has been measured by LDA. The change in the basic state of the flow can be inferred from heat-transfer measurements. Such experiments have been carried out by Dybbs and Edwards at Case Western.

T. Wu: Can you distinguish whether oil is of mineral or biological origin?

Schwartz: I have been told that oil's immediate precursor is keragen, the hydrocarbon component found in oil-shale, for example.

Van Hooff: It is still an open question as to the original process of oil formation. Regardless of the process of formation, in order for hydrocarbons to be produced, we still need a reservoir with cor-rock.