# ENERGY CALCULATIONS BASED ON THE MODIFIED ZAKHAROV EQUATION

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## 1. INTRODUCTION

The equations governing the irrotational flow of an incompressible inviscid fluid with a free surface and infinitely deep bottom can be transformed into an evolution equation in the Fourier-plane:

$$i\,\frac{\partial B}{\partial t} = I_3(B,\underline{k},t) + I_4(B,\underline{k},t) + \ldots \qquad (1)$$

The dependent variable $B(\underline{k},t)$ represents the free-components of the wave-field. $I_3$, $I_4\ldots$ , are integral operators representing quartet, quintet, ..., nonlinear interactions, respectively.

The leading term on the r.h.s. of (1) was first derived by Zakharov in 1968 and the higher-order term $I_4$ was obtained in Stiassnie & Shemer (1984).
$B(\underline{k},t)$ is related to the Fourier-transform (denoted by a hat) of the free surface elevation $\eta(\underline{x},t)$ and of $\phi^s(\underline{x},t) = \phi(\underline{x},\eta(\underline{x},t),t)$ - the velocity potential at the free surface, through $b(\underline{k},t)$ - which is a kind of generalized 'amplitude' spectrum.

$$\hat{\eta}(\underline{k},t) = \left(\frac{\omega}{2g}\right)^{\frac{1}{2}}[b(\underline{k},t)+b^*(-\underline{k},t)],\quad \omega(\underline{k}) = (g|\underline{k}|)^{\frac{1}{2}}, \qquad (2a)$$

$$\hat{\phi}^s(\underline{k},t) = -i\left(\frac{g}{2\omega}\right)^{\frac{1}{2}}[b(\underline{k},t) - b^*(-\underline{k},t)], \qquad (2b)$$

and

$$b(\underline{k},t) = [B + B'+B''+B'''+\ldots]\, e^{-i\omega(\underline{k})t} \qquad (3)$$

The quantities $B'$, $B''$,... represent second order, third order,... locked-components of the wave field, all known functions of $B$, $\underline{k}$, and $t$.

In the present study we use a discretized version of (1) and a Runge-Kutta method to calculate the evolution of wave fields which consist of at most five free components:

$$\eta = \sum_{n=1}^{5} a_n \cos(\underline{k}_n\cdot\underline{x} - \int_0^t \Omega_n\, dt+\theta_n) + \text{higher order locked components} \qquad (4)$$

The wave (1) is the leading component of a Stokes wave - sometimes called 'the carrier'. The two couples, (2,3) and (4,5) were chosen to be the most unstable disturbances of class I and class II instabilities respectively. These disturbances were given an initial amplitude equal to 10% of the carrier-amplitude.
As in Shemer & Stiassnie (1985) we look again at two simpler cases: The first deals with class I instabilities only ($B_4 \equiv B_5 \equiv 0$), and the second is restricted to class II instabilities ($B_2 \equiv B_3 \equiv 0$).

## 2. ENERGY CALCULATIONS

The exact equations of motion for water-waves are known to form a Hamiltonian system, and the total energy of the entire wave-field is conserved. The average energy density, taken over the $(x_1,x_2)$ plane is given by:

$$h = \lim_{L\to\infty} \frac{1}{(4L^2)} \int_{-L}^{L} \int_{-L}^{L} \tfrac{1}{2}(g\eta^2 + \phi^s \frac{\partial \eta}{\partial t}) dx_1 dx_2 \tag{5}$$

Any exact solution should give h = constant, for all t. When a truncated version of (1) is used, one can expect (5) to yield h(t) which is only approximately constant.

The accuracy of the calculated h is related to the accuracy of the 'amplitudes' b. To obtain h accurate to order $(a_1k_1)^2$, b should be accurate to order $(a_1k_1)$ thus only the free waves are required.

A more detailed investigation shows that the result obtained by using only the free-waves is accurate to order $(k_1a_1)^3$ and has an error of order $(k_1a_1)^4$. We denote this result by $h_3$.

$$h_3 = \frac{1}{4\pi^2} \sum_{n=1}^{5} \omega_n |B_n|^2 = \frac{g}{2} \sum_{n=1}^{5} a_n^2 \tag{6}$$

For higher order corrections one has to include the locked waves. To obtain $h_4$ (h accurate to order $(a_1k_1)^4$) 580 wave components are required; these include B, B', B" and yield products of B with B", and B' with B'. $h_5$ is obtained when all the 3705 wave components (5 free and 3700 tuged) are included; thus adding products of B with B'" and B' with B", Note that in order to obtain an accuracy higher than $h_5$ one has to add higher order terms on the r.h.s. of (1).

## 3. RESULTS

In Fig. 1 we show the variation with time of the amplitudes of the free waves for class I instability (in the upper row), class II instability (in the middle row) and for the coupled instability (in the lower row). The results are for three different Stokes waves having initial steepness: $\varepsilon = 0.130$ (in the left column), $\varepsilon = 0.227$ (middle column), and $\varepsilon = 0.336$ (right column). The curve (1) is for the carrier amplitude, the curves (2), (3) are those for the amplitudes of the most unstable class I disturbances and the curves (4), (5) which coalesce for the present problem, are for the most unstable class II disturbances. All nine figures have a duration of about 400 carrier periods.

The uppermost curves in Fig. 1 represent three approximations of the average energy density, i.e. $h_3$, $h_4$ and $h_5$.

Class I interaction: one can see that contribution of the energy terms of the order $(k_1a_1)^4$ leads to a considerable improvement in the conservation of the calculated energy in the evolution process, and $h_4$ does not deviate practically from a horizontal curve, with the exception of the highest amplitude considered. The higher order, $h_5$, curve does not differ from $h_4$.

Class II interaction: The middle row of Fig. 1 shows that the addition of the energy terms of the order $(k_1a_1)^4$ changes only the 'mean level' of the energy density. In order to obtain improvement in the energy conservation one has to take into account higher order, $(k_1a_1)^5$ terms. Note that these terms are not necessarily positive and the $h_4$ and $h_5$ curves intersect.

Coupled (Class I + Class II) interaction: For the lowest amplitude considered ($\epsilon$=0.13) the curves $h_4$ and $h_5$ hardly differ from each other and from the horizontal straight line, giving an improvement compared to $h_3$. For $\epsilon$ = 0.224, $h_4$ and $h_5$ give considerably better results than $h_3$ , but the deviations from a horizontal line are clearly seen. At even higher amplitude $\epsilon$ = 0.336, one sees that the present order of approximation is not sufficient. Note that the conclusion of Yuen & Lake (1982) p. 196 that the Zakharov approximation does not conserve energy stems from the fact that they refered to $h_3$, and did not take into account the higher order approximation $h_4$. We believe that the above results indicate that the modified Zakharov is a consistent approximation of the water-wave problem.

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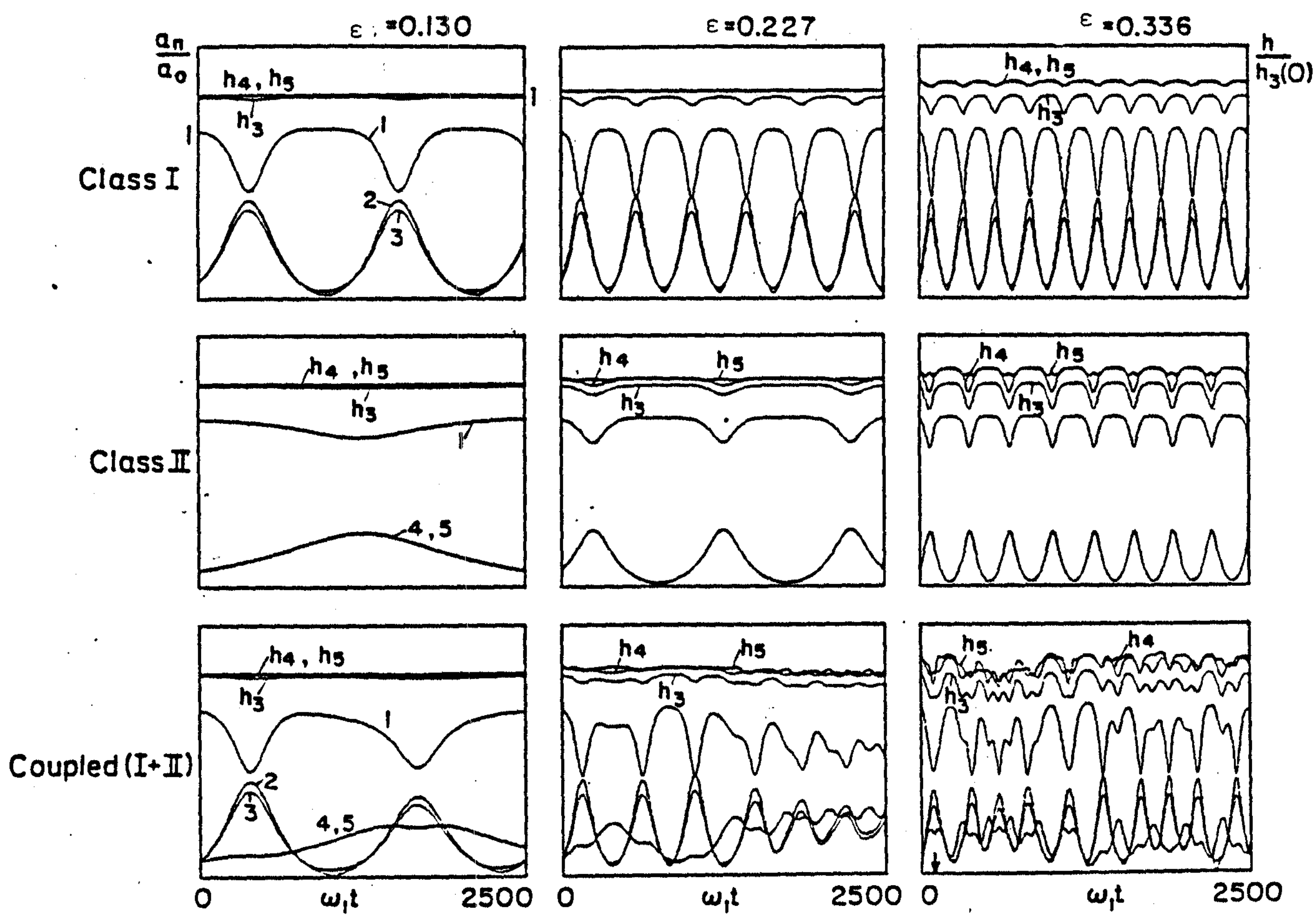


Fig. 1: Dependence of evolution process on the carrier steepness

## Discussion

Schwartz: Linear instability is only applicable for short times. Here you trace the time evolution. What is the ultimate fate of all of these instabilities?

Stiassnie: The ultimate fate of these instabilities depends on the initial 'noise level' or the initial disturbances. In some cases, it leads to Fermi-Pasta-Ulam recurrence, in others to an aperiodic (maybe chaotic) behavior. For very steep waves the situation is even more complex due to the occurrence of local wave-breaking.