Causality and the Radiation Condition

by
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We wish to consider here one aspect of the motion of a body floating in a heavy fluid. The motions will be assumed to be small enough so that the equations may be linearized. In order to describe the motion of both body and fluid, we shall adopt a right-handed coordinate system with Oz directed against gravity, Ox to the right, and Oy into the paper. The plane Oxy lies in the undisturbed free surface. The small excursions that the body makes about its fixed equilibrium position will be denoted by $\alpha_1, \cdots, \alpha_6$, where $\alpha_1, \alpha_2, \alpha_3$ represent translational displacements and $\alpha_4, \alpha_5, \alpha_6$ angular ones. The dynamical constants of the body will be denoted by m_{jk} where $m_{11}=m_{22}=m_{33}=m$, the mass of the body, and

$$m_{jk} = \int \rho [r^2 \delta_{jk} - x_j x_k]\, dV, \quad j,k = 4, 5, 6,$$

where ρ is the density distribution of the body, $r^2 = x_i x_i$, and the integral is taken over the body. All other m_{jk} are zero. In the equations to be given below c_{jk} are the hydrostatic coefficients, μ_{jk} are the added masses as defined by Cummins(1962) (i.e., the added masses at infinite frequency), and $L_{jk}(t)$ is a weighting function defined in terms of the velocity potential for the fluid motion. Its definition as well as those of μ_{jk} and c_{jk} may be found in Wehausen(1971 or 1967). An important property of L_{jk} is that it is zero for $t<0$. Let $X_j(t)$ be the force ($j=1,2,3$) or moment ($j=4,5,6$) to which the body is subjected. $X_j(t)$ may be a result of oncoming waves, of wind, or, for example, of some exterior forcing mechanism. We suppose it to be absolutely integrable. Then the linearized equations of motion for the body are as follows (see Wehausen, loc. cit.):

$$(m_{jk} + \mu_{jk})\ddot{\alpha}(t) + c_{jk}\alpha_k + \int_{-\infty}^{t} L_{jk}(t-\tau)\,\ddot{\alpha}_k(\tau)\, d\tau = X_j(t).$$

The equations are a natural candidate for a Fourier (or Laplace) transform. We shall use the Fourier transform:

$$X_k(t) = \int_{-\infty}^{\infty} \tilde{X}_k(\sigma)\, e^{-i\sigma t}\, d\sigma.$$

After taking the Fourier transform of the equations of motion, we find the following:

$$\{-\sigma^2[m_{ik} + \mu_{ik}(\sigma)] + c_{ik} - i\sigma\lambda_{ik}(\sigma)\}\,\tilde{\alpha}_k(\sigma) = \tilde{X}_i(\sigma),$$

where

$$\mu_{ik}(\sigma) - \mu_{ik}(\infty) + i\sigma^{-1}\lambda_{ik}(\sigma) = \int_0^\infty L_{ik}(\tau)\, e^{i\sigma\tau}\, d\tau.$$

It follows from this that $\mu_{ik}(-\sigma) = \mu_{ik}(\sigma)$ and $\lambda_{ik}(-\sigma) = \lambda_{ik}(\sigma)$. We introduce the following notation:

$$M_{ik} = -\sigma^2[m_{ik} + \mu_{ik}] + c_{ik},\ N_{ik} = \sigma\lambda_{ik},\ \tilde{S}_{ik} = M_{ik} - iN_{ik}.$$

The transformed equation then reads:

$$\tilde{S}_{ik}\,\tilde{\alpha}_k = \tilde{X}_i,$$

and its solution is evidently

$$\tilde{\alpha}_i = \tilde{T}_{ik}\,\tilde{X}_k,$$

where $\tilde{T} = \tilde{S}^{-1}$, i. e. $\tilde{T}_{ij}\tilde{S}_{jk} = \delta_{ik}$. It now follows easily from the above and from known properties of L_{ik} that $\tilde{S}_{ik}(-\sigma) = \bar{\tilde{S}}_{ik}(\sigma)$, $\tilde{S}_{ki} = \tilde{S}_{ik}$ and similarly for $\tilde{T}_{ik}$. Because of this property of $\tilde{T}_{ik}$ we find

$$\begin{aligned} T_{ik}(t) &= (2\pi)^{-1}\int_{-\infty}^{\infty} \tilde{T}_{ik}(\sigma)\, e^{-i\sigma t} d\sigma \\ &= (2\pi)^{-1}\int_0^\infty [\bar{\tilde{T}}_{ik}(\sigma)\, e^{i\sigma t} + \tilde{T}_{ik}(\sigma)\, e^{-i\sigma t}]\, d\sigma, \end{aligned}$$

i. e. $T_{ik}(t)$ is real.

Having found $\tilde{\alpha}_i(\sigma)$ above, we may now calculate $\alpha_i(t)$:

$$\begin{aligned}\alpha_i(t) &= \int_{-\infty}^{\infty} \tilde{\alpha}_i(\sigma)\, e^{-i\sigma t}\, d\sigma = \int_{-\infty}^{\infty} \tilde{T}_{ik}\tilde{X}_k\, e^{-i\sigma t}\, d\sigma \\ &= (2\pi)^{-1}\int_{-\infty}^{\infty} \tilde{T}_{ik}\, e^{-i\sigma t}\, [\int_{-\infty}^{\infty} X_k(\tau)\, e^{i\sigma\tau}\, d\tau]\, d\sigma \\ &= (2\pi)^{-1}\int_{-\infty}^{\infty} d\tau\, X_k(\tau)\int_{-\infty}^{\infty} d\sigma\, \tilde{T}_{ik}(\sigma)\, e^{-i\sigma(t-\tau)} \\ &= (2\pi)^{-1}\int_{-\infty}^{\infty} X_k(\tau) T_{ik}(t-\tau)\, d\tau \\ &= (2\pi)^{-1}\int_{-\infty}^{\infty} T_{ik}(\tau) X_k(t-\tau)\, d\tau\,.\end{aligned}$$

The preceding development is well known and has been reviewed in order to display the last formula for $\alpha_i(t)$, for this formula seems to indicate that α_i at time t depends upon the value of the exciting force X_k at all future as well as all past times unless we can show that $T_{ik}(t) = 0$ for all $t<0$. It is this problem that we wish to address and to which we now turn.

First we shall show that the desired property of T_{ik} is equivalent to a certain property of $\tilde{T}_{ik}$. The reasoning is well known and can be found in books on control theory (e. g., Solodovnikov, 1960, pp.24-28). Consider the transform

$$\tilde{T}_{ik}(\sigma) = (2\pi)^{-1}\int_{-\infty}^{\infty} T_{ik}(t)\, e^{i\sigma t}\, dt\,.$$

Although heretofore we have thought of σ as being real, we shall now take it to be complex. $\tilde{T}_{ik}(\sigma)$ is then defined in the whole σ-plane. Let us write $\sigma = \rho e^{i\theta} = \rho(\cos\theta + i\sin\theta)$. Consider now

$$\int \tilde{T}_{ik}(\sigma)\, e^{-i\sigma t} d\sigma = \int \tilde{T}_{ik}(Re^{i\theta})\, e^{-iRt\cos\theta}\, e^{Rt\sin\theta}\, Rie^{i\theta}\, d\theta\,,$$

where the path of integration is either along the semicircle C_+: $\rho = R$, $0<\theta<\pi$, or the semicircle C_-: $\rho = R$, $2\pi>\theta>\pi$. These paths are now completed by paths along the real axis from $-R$ to R. Evidently, as $R\to\infty$ the integral along C_+ converges to zero if $t<0$ and that along C_- converges to zero if $t>0$. It then follows that for $t<0$

$$T_{ik}(t) = \lim_{R\to\infty} [\int_{-R}^{R} \tilde{T}_{ik}(\sigma)\, e^{-i\sigma t} d\sigma + \int_{C_+} \tilde{T}_{ik}(\sigma) e^{-i\sigma t}\, d\sigma].$$

If $\tilde{T}_{ik}$ is analytic in the upper half-plane, then $T_{ik}(t) = 0$ for $t<0$. The converse of this is also true, i. e. if $T_{ik}(t) = 0$ for $t<0$, then $\tilde{T}_{ik}(\sigma)$ is analytic in the upper half-plane.

Our problem has now been transformed to that of showing that $\tilde{T}_{ik}$ is analytic in the upper half-plane. Since $\tilde{T} = \tilde{S}^{-1}$, we shall search for an equivalent property of $\tilde{S}$. Let P_{ik} be the cofactor of the element $\tilde{S}_{ik}$. Then it is known that

$$\tilde{T}_{ik} = P_{ki}/\det \tilde{S}.$$

As we shall see, $\tilde{S}$ and hence P_{ik} are analytic in the upper half-plane. Thus what remains to be shown is that $\det \tilde{S}$ has no zeros in the upper half-plane. How do we know that $\tilde{S}$ is analytic in the upper half-plane? From the earlier formula defining μ_{ik} and λ_{ik} it follows that

$$\tilde{S}_{ik} = -\sigma^2 \int_0^\infty L_{ik}(t)\, e^{i\sigma t} dt + c_{ik} - \sigma^2 [m_{ik} + \mu_{ik}(\infty)].$$

It has already been mentioned that $L_{ik}(t) = 0$ for $t<0$, so that its transform is analytic in the upper half-plane. Since the other terms in the equation above are obviously analytic, $\tilde{S}_{ik}$ is also.

We turn now to $\det \tilde{S}$. The matrix $\tilde{S} = M - iN$ is symmetric but is not hermitian, so that no easy conclusion can be drawn from the fact of symmetry alone. However, the matrix

$$\tilde{S}\overline{\tilde{S}} = (M - iN)(M + iN) = M^2 + N^2 + i(MN - NM)$$

is hermitian, and we shall be able to exploit this fact.

Associated with any hermitian matrix $A_{ik} = \overline{A}_{ki}$, $i,k=1,\cdots,n,$ is a so-called hermitian form

$$Q = x_i A_{ik} \overline{x}_k ,$$

where repeated indices are to be summed from 1 to n. It is easy to show that $\bar{Q} = Q$, so that Q is real. Within this class of forms one distinguishes positive (negative) definite and non-negative (non-positive) definite forms. A non-negative definite form is one such that $Q \geq 0$ for any choice of $x_1, \cdots, x_n$; a positive definite form is one such that $Q > 0$ for any choice of $x_1, \cdots, x_n$ except $x_1 = x_2 = \cdots = x_n = 0$. Analogously for the terms in parentheses. A classic theorem about hermitian forms states that such a form is positive definite if and only if all the determinants formed with the first minors along the main diagonal are positive, i. e.,

$$A_{11} > 0, \quad \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} > 0, \quad \cdots, \quad \begin{vmatrix} A_{11} & \cdot & \cdot & \cdot & A_{1n} \\ A_{21} & \cdot & \cdot & \cdot & A_{2n} \\ \cdot & & & & \cdot \\ A_{n1} & \cdot & \cdot & \cdot & A_{nn} \end{vmatrix} > 0 .$$

It then follows that *all* main-diagonal minors are positive. There is an analogous theorem for negative definite forms and a somewhat more complicated one for non-negative and non-positive definite forms. However, the only part of these theorems that we shall need is the statement that $\det A > 0$ if Q is positive definite.

Consider now the special hermitian form

$$\begin{aligned} Q &= x_i(M_{ij} - iN_{ij})(M_{jk} + iN_{jk})\bar{x}_k = x_i \tilde{S}_{ij} \bar{\tilde{S}}_{jk} \bar{x}_k \\ &= x_i(M_{ij} - iN_{ij}) \overline{\bar{x}_k(M_{kj} + iN_{kj})} \\ &= x_i(M_{ij} - iN_{ij}) \overline{x_k(M_{kj} - iN_{kj})} \\ &= \textstyle\sum_j | x_i(M_{ij} - iN_{ij})|^2. \end{aligned}$$

Evidently Q is a sum of squares and hence $Q \geq 0$. But Q can $=0$ for a particular 6-tuple $x_1, x_2, \cdots, x_6$ if and only if

$$x_i(M_{ij} - iN_{ij}) = 0 , \quad j = 1, \cdots, 6.$$

If one now multiplies by $\bar{x}_j$ and sums, one finds

$$x_i(M_{ij} - iN_{ij})\bar{x}_j = 0 \quad \text{or} \quad x_i M_{ij} \bar{x}_j = i\, x_i N_{ij} \bar{x}_j .$$

Since both M_{ij} and N_{ij} are real, the forms $x_i M_{ij} \bar{x}_j$ and $x_i N_{ij} \bar{x}_j$ are also real. But then both must be zero for this particular 6-tuple. This, however, is not possible for $x_i N_{ij} \bar{x}_j$ unless all $x_i = 0$, for this form is known to be positive-definite, a consequence of the radiation condition [see, e.g., Wehausen, 1971, p.245, eq. 26]. Hence the form Q must be positive-definite and consequently the determinant

$$\det \tilde{S}\bar{\tilde{S}} = \det \tilde{S} \cdot \det \bar{\tilde{S}} > 0 \quad \text{or} \quad \det \tilde{S} \neq 0.$$

Retracing our steps, we see that if the radiation condition is satisfied then $T(t) = 0$ for $t < 0$ and hence that the future does not determine the present, at least for linearized water-wave theory.

One may note that it is not necessary for M_{jk} to be positive-definite or even non-negative definite. It is known that m_{jk} is positive definite, that c_{jk} is non-negative definite, and that μ_{jk} is neither. However, none of this information is relevant. It is the radiation condition, i. e. the condition that energy is carried away from an oscillating body, that allows us to prove that $T(t) = 0$ for $t < 0$.

This topic is discussed in an earlier paper (Wehausen, 1971), but the treatment there is inadequate and incomplete and hardly does more than pose the problem.

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Discussion

Tuck: Could you explain why the original weighting function $L_{ij} = 0$ for $t < 0$? I would have thought that implies causality.

Wehausen: I think that the answer is "no". $L_{ij} = 0$ for $t < 0$ is a consequence of the formulation of an initial-value problem and the definition of L_{ij} in terms of the time-dependent velocity potential. $T_{ij}(t) = 0$ for $t < 0$ is a consequence of the radiation condition. However, one must admit that the whole problem arises as a consequence of taking the Fourier transform of the original problem and then the inverse transform. Perhaps there is something artificial here.

Evans: There are cases when $N_{ij} = 0$ at certain frequencies, for example for axisymmetric bodies in heave which are bulbous below the surface. Does this affect the causality proof? Is it possible to show $L_{ij} = 0$ for $t < 0$?

Wehausen: Since these frequencies are isolated, I would conjecture that they play no role in the behavior of the function

$$T_{ij}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{T}_{ij}(\sigma)\, e^{i\sigma\tau}\, d\tau$$

Ursell: The most common method of treating transient problems is by Laplace transforms (or one-sided Fourier transforms). Thus if we assume that the potential $\phi(x,y,z,t)$ is bounded for all x, y, z and t, it will then follow that the transform $\Phi(x,y,z,\omega) = \int_0^\infty \phi(x,y,z,t)e^{i\omega t}dt$ is an analytic function of in the upper half-plane Im $\omega > 0$, and that Φ is bounded for fixed ω when $x^2+y^2+z^2 \to \infty$. It can be shown that Φ satisfies the equations for periodic oscillations of frequency ω (see e.g. J. Fluid Mech. 19, 1964, p. 309) and because of our boundedness assumption it can also be shown that Φ satisfies the usual radiation condition. My difficulty is this: How do we show that $\phi(x,y,z,t)$ is bounded? This difficulty occurs very often when we seek to apply Laplace transforms but has never (I believe) been resolved. Is this the same difficulty discussed in the present work, or is it quite separate?

Wehausen: It may very well be the same problem, or perhaps a small aspect of it. Essentially, all that I am proving is that if the radiation condition is satisfied then $T_{ij}(t) = 0$. I would be surprised if it entailed the boundedness of $\phi(x,y,z,t)$

Yeung: Embedded in the proof is essentially the absence of poles of the transfer function $\tilde{T}(\sigma)$ in the upper half plane. If the domain is a closed one, it would seem the positive definitness of λ_{ij} cannot be assured. Hence a possible corollary of the proof may be that T(t) will depend on the future. This however does not seem so physically plausible, does it?

Wehausen: I have not considered the closed-domain problem. However, $x_i \lambda_{ij} \tilde{x}_j > 0$ is a consequence of $\int_\Sigma \rho \phi_t \phi_n \, dS > 0$ for the unbounded fluid. For a fluid contained in a closed basin, this integral evidently vanishes on the basin wall and no energy is being lost. I will look at the question later.

Kleinman: Is there an implicit assumption that L are smooth functions of time?

Wehausen: I have not considered such questions, but I would conjecture that smoothness is a consequence of the definition of L_{ik}.

Van Hooff: This discussion reminds me of Routh-Hurwitz stability criterion for O.D.E.'s. In that case, stability is directly associated to causality.

Wehausen: I do not see the connection, but one may exist.

Newman: Implicitly, the proof assumes the forward speed U = 0. For U > 0, various exceptions may arise including negative damping, yet we believe that causality still applies. Have you considered this?

Wehausen: When $U \neq 0$, it is known that $\mu_{ij} \neq \mu_{ji}$ and $\lambda_{ij} \neq \lambda_{ji}$ in general. The proof in the paper seemed to rely upon symmetry of the matrix S_{ij}. However, this is not necessary. The product $S\bar{S}^T$ is Hermitian. The condition $\int_\Sigma \rho \phi_t \phi_n \, dS > 0$ leads to the positive-definiteness of the quadratic form

$$x_i \left[\sigma(\lambda_{ij} + \lambda_{ji}) + i\sigma^2(\mu_{ij} - \mu_{ji})\right] \bar{x}_j > 0$$

A development similar to that in the paper leads to the positive definitiveness of the Hermitian form $x_i S_{ik} \bar{S}_{jk} \bar{x}_j$ and thence back to the conclusion $T(t) = 0$ for $t < 0$.

T. Wu: Professor Wehausen, with this contribution of yours, it seems possible to establish a general rule for imposing the physically appropriate radiation condition. If established, it should be a very valuable achievement because such a condition for the most general case can be difficult, even for the case of conventional formulation, leaving alone further variations and nonlinear theory. (For example, consider the case of an obstacle, moving forward while oscillating in some modes in a free stream that is sheared. Would a valid radiation condition be readily available, under both gravity and capillary effects?)

In regard to Professor Ursell's comments, I fully agree that in considering the initial-value problem (using the Laplace transform on linear theory, for instance, or using the time-domain approach for nonlinear problems of free-surface waves) it is always possible to curtail the radiation condition and one can always go directly to the large time asymptotic result (based on linear theory) provided the initial data is sufficiently limited in spatial distribution so that the necessary and sufficient conditions are satisfied in applying the Tauberian theorem.

Evans: N_{ij} may be negative without forward motion in certain cases. Does your proof breakdown at these frequencies?

Wehausen: Individual N_{ij}'s may be negative, but the quadratic form $x_i\, N_{ij}\, \overline{x}_j$ must be positive, and this is all that is required.

Evans: Wehausen has opened up an area which we have glossed over for some time. I hope this will inspire much discussion.