On the Application of the Boundary Element Method to Two-Dimensional Free-Surface Interactions with Bodies

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The Boundary Element Method (BEM) is now commonly used as an efficient and accurate numerical technique in many applications in engineering practice and mathematical physics. It is ideally suited to the study of the free-surface flow problems and those of floating bodies. The method, originally devised for the solution of linear two- and three-dimensional potential and elasticity problems, has now been refined and extended to cover more complicated ones such non-linear, time-dependent and transient problems. The method consists of the transformation of the governing partial differential equations for the unknown within and on the region of interest to an integral equation over the boundary of the domain, and the solution of these equations for functions on the boundary alone. If values at interior points are required, they are calculated afterwards from the boundary data. The advantage of using this form of numerical formulation is that it can also deal with three-dimensional problems. The boundary integral equation method based on the Cauchy integral theorem cannot be extended to three-dimensional problems since the complex potential function exists only in a two-dimensional space.

This contribution presents the point collocation BEM for two-dimensional potential problems with particular attention to the analysis of non-linear water wave problems and the study of the interaction between the free surface and solid boundaries. The problems are formulated mathematically as two-dimensional, non-linear, initial-boundary value problems in terms of a velocity potential, assuming the fluid to be inviscid and incompressible, and the flow to be irrotational.

The integral equation is formulated through the application of Green's third identity which represents an harmonic function as the superposition of a single-layer and a double-layer potential. Taking the field point to the boundary, an integral equation relating only boundary values and normal derivatives of the harmonic function is obtained.

Let $\phi(p)$ and the function $G(p,q)$ represent the unknown harmonic function and the fundamental solution to Laplace's equation respectively. The boundary integral equation is thus

$$2\pi\phi(p) + \int_{\Gamma} \phi(Q)\frac{\partial G(p,Q)}{\partial n(Q)}\, d\Gamma(Q) = \int_{\Gamma} \left.\frac{\partial \phi}{\partial n}\right|_{Q} G(p,Q)\, d\Gamma(Q)$$

Let the interior point p(x) be taken to a boundary point

P(x). The function $\phi(p)$ is assumed continuous so that $\phi(p) > \phi(P)$. The second integral in the above equation is also continuous if $\partial\phi/\partial n$ is bounded. The first integral contains a double-layer potential with density $\phi(Q)$ and this potential has a jump of $-\pi$ in going from p(x) to P(x). Thus, in the limit, this equation becomes,

$$\pi\phi(P) + \int_\Gamma \phi(Q)\frac{\partial G(P,Q)}{\partial n(Q)}\, d\Gamma(Q) = \int_\Gamma G(P,Q)\left.\frac{\partial\phi}{\partial n}\right|_Q d\Gamma(Q)$$

This equation states that a constraint equation between the Dirichlet boundary conditions and the Neumann boundary conditions must be satisfied for all harmonic functions. This constraint equation is the so-called boundary integral equation which is the basis of this work. If the solution to the Neumann problem is desired, the right-hand side of this equation is known and a Fredholm equation of the second kind for the unknown boundary values of the function, $\phi(Q)$, is obtained. If $\phi(P)$ is known, i.e. the Dirichlet problem, then this equation becomes a Fredholm equation of the first kind for the unknown boundary values of $\partial\phi/\partial n$. The mixed boundary value problem, i.e. the Cauchy problem, leads to a mixed integral equation for the unknown boundary data.

For the numerical implementation, the variations of ϕ and $\partial\phi/\partial n$ are assumed to be linear within each element. The use of such elements has proved to be stable numerically for certain non-linear water wave problems and for the cases to be presented it was not necessary to use any smoothing procedure since there was no detectable growth of numerical instability. Higher order elements, however, were found to exhibit instability and these had to be suppressed by the use of smoothing functions. One apparent disadvantage of the BEM over the other numerical techniques, notably the finite element method, is the inherent presence of singularities in the numerical integration. However, this is not a major problem as the numerical techniques for evaluating the singular and non-singular boundary integrals have been developed. When the boundary Γ of the boundary integral domain Ω has one or more geometrical corners, problems are also encountered owing to the discontinuous nature of the normal derivatives $\partial\phi/\partial n$. Such problems also exist at the fluid-body intersections, but techniques have been developed to give accurate representations in such regions.

For the time-stepping procedure, fluid particles on the free surface are followed. The position and the velocity potential of these points are obtained by integrating in time the kinematic free-surface boundary condition which stipulates that the free surface is a material surface. The advantage of using the Lagrangian description for the fluid motion is that the position of the free surface is known. This means that when solving the problem of free surface flow, only the equations, and not the boundary, need to be approximated. However, for steady-state flow and that involving a small uniform current in a transient flow, the semi-Lagrangian scheme for the description of the fluid motion, where particles are restricted to vertical movement, is deemed necessary so that particles do not all translate downstream. The development of the flow is obtained by an explicit time-stepping procedure in which the flow at each time step is calculated by the BEM. Three different time-stepping schemes were used and the only difference between them was the time taken for each iteration or time-step. For equivalent solutions using linear elements, otherwise identical calculations took 1.91 CPU seconds on the CDC Cyber 205 for the second-order Euler predictor and Runge-Kutta corrector scheme, 2.1 seconds for the fourth-order Adams-Bashforth predictor and modified Adams-Moulton corrector method and 1.89 seconds per time-step for the truncated Taylor series. "Incomplete" solutions, whereby the boundary-integral kernels are unchanged (assuming a nearly flat surface), were also used because they were numerically efficient. The respective figures for such runs were 0.41, 0.47 and 0.81 seconds.

Simple-harmonic flows using a wavemaker similar to that used by Lin et al (1984) were investigated in order to assess the numerical method and its stability characteristics. Without any smoothing, results for the non-linear motions were in reasonable agreement with other numerical solutions and equivalent experimental observations.

Impulsively-started flows were investigated as a prelude to the study of wedge-entry problems. The numerical results based on the BEM were compared with Peregrine's analytical solution and were found to be satisfactory for discretisation of the boundary based on constant element spacing on the free surface. Agreement was improved for discretisation based on an exponential spacing on the free surface giving smaller element spacing close to the intersection, although the solution became divergent earlier as the local element size was decreased. This was thought to be due to the problem of matching either a dynamic or kinematic boundary condition at the intersection between the wavemaker and the fluid surface. Similar problems were also encountered with wedge entry into (initially) calm water. For a half-apex angle of up to 30°, good comparisons were obtained with known analytical results for zero gravity. However, beyond that angle, problems with numerical instability and slow convergence became severe. It would appear that the potential-flow approximations give good results at a small distance from the intersection between the wedge and the free surface but not at the intersection itself. It is felt that the potential-flow assumption is an over-simplification at this point.

The problem of a floating body of radius a and incident wave amplitude A was also dealt with for A/a ratios up to 0.8 where substantial non-linear effects are clearly present. It was found that reasonable agreement with "reliable" experimental results was obtained for a solution using the fully Lagrangian description of fluid motion. The variations of wave frequency and the body dimensions were also investigated.

A small uniform current was subsequently introduced and its effects were studied. For this case, a semi-Lagrangian scheme was used. Initial velocity potential values and free surface profiles were obtained from an iterative procedure giving steady-state results and the wavemaker at one end was then imposed as before.

Reference

LIN, W.-M., J.N. NEWMAN & D.K. YUE (1984): "Nonlinear forced motions of floating bodies", 15th Symp. Naval Hydrodynamics, Hamburg, West Germany.