On the Analytic Form of Wave Solutions in the Frequency Domain

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In this work the classical approach to solving the hydrodynamic interaction between waves and structures is considered, namely by a decomposition of the linearised potential problem into monochromatic radiation and diffraction problems. There are well-established numerical procedures which yield the hydrodynamic quantities of interest; however, they give no information about the analytic form of these quantities as functions of the dimensionless wavenumber. The asymptotic results which are available at large or small wavenumbers are not accurate in the 'middle range', that is when the body dimension is close to the wavelength. There is, however, one analytic result, due to Ursell(1963), which will be used here to yield numerical results.

Ursell proved that, for a heaving hemisphere, the complex radiation impedance $\Lambda(\mu) = A + iB$ is given by

$$\Lambda(\mu) = \frac{E_1(\mu) + E_2(\mu)\,L}{E_3(\mu) + E_4(\mu)\,L} \qquad (\,L = \ln\mu + \gamma - i\pi\,)$$

where μ = dimensionless wavenumber, A = added mass, B = damping, γ is Euler's constant and the four functions E_i are entire and real for real μ. This form is purely a consequence of the decomposition of the Green function

$$G(\underline{r},\underline{r}';\mu) = G_1(\underline{r},\underline{r}';\mu) + G_2(\underline{r},\underline{r}';\mu)L$$

(where G_1 & G_2 are real for real μ, and have power-series expressions which converge for all μ), and so the analytic form of Λ should apply to a wide class of two- and three-dimensional bodies, and to different modes of motion.

Ursell achieved this result by normalising the source term in the potential, rather than the velocity in the boundary condition on the body. In the context of the present work that suggests writing the body boundary condition as

$$\frac{\partial\phi}{\partial n} = (\,V_1(\mu) + V_2(\mu)L\,)n_y$$

for heave, and looking for functions V_1 & V_2 such that the potential ϕ can be expressed as

$$\phi = \phi_1(\underline{r};\mu) + \phi_2(\underline{r};\mu)L$$

where the potentials ϕ_1 & ϕ_2 are real for real μ and entire in μ . (Note that it is not possible to solve

$$\frac{\partial\phi_i}{\partial n} = V_i(\mu)n_y \qquad (\,i = 1,2\,)$$

as the radiation condition needed only applies to the whole of ϕ).

The problem for ϕ_1 and ϕ_2 is solved using two integral equations with the same kernel, and then a third equation yields the value V_2/V_1 (= E_4/E_3). The choice V_1= 1 is useful if only small values of μ are being considered, but more generally choosing V_1 to equal the Fredholm determinant of the integral equations may be more appropriate. Computed results for Λ for the heaving semi-circular cylinder are identical to those achieved by other methods, with the advantage that the functions $(E_i(\mu)/E_3(\mu))$ (i=1,2 & 4) are available numerically from this method.

The above method requires two real integral equations in place of the single complex integral equation in the conventional approach. For symmetric bodies these two real equations can be inverted together because of the simple form of G_2; furthermore, this approach constitutes a proof of the analytic form of Λ for general symmetric bodies, or for axisymmetric bodies in three dimensions.

All the above can also be extended to deal with the diffraction problem, where a third integral equation (which can be inverted with the other two) gives sufficient additional information for the method to yield the complex exciting force F which has the same analytic form as Λ . In fact, it is easily shown that E_4/E_3 has the same value for F as for Λ .

Reference:

Ursell, F. 1963 "The periodic heaving motion of a half-immersed sphere: the analytic form of the velocity potential and the long-wave asymptotics of the virtual mass" Dept. of Mathematics Internal Rep. , Univ. Manchester.