Trapping modes in the theory of surface waves

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At the First International Workshop there was a paper on trapping modes by Aranha. He considers irrotational motions, with velocity potentials of the form

$$\varphi(x,y) \cos kz\, e^{i\omega t},$$

where $\varphi(x,y)$ satisfies the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2\right) \varphi(x,y) = 0$$

in a domain D which is bounded above by the mean free surface $y = 0$ and which is bounded internally by a curve C not intersecting $y = 0$. The boundary conditions are $\frac{\partial\varphi}{\partial y} = -\frac{\omega^2}{g}\varphi = -K\varphi$ when $y = 0$, and $\frac{\partial\varphi}{\partial n} = 0$ on C.

It was one of Aranha's aims to prove the mathematical existence of trapping modes, i.e., of motions which tend rapidly to 0 at ∞. Such motions can exist only for certain eigenvalues $K_1, K_2, \ldots$ depending on C. I am not yet convinced that Aranha's hybrid formulation can lead to a proof of the existence of trapping modes, but proofs have previously been given by Ursell (1951) for a small circle and by D.S. Jones (1953) for an arbitrary curve C with a vertical line of symmetry. Here a new and simpler method is proposed. The parameter K is treated as the eigenparameter and k^2 is treated as known. (Jones treated k^2 as the eigenparameter and K as known). Then an integral equation is obtained for $v(x)$, the vertical velocity component on $y = 0$. This integral equation is of the form $v(x) = K\int_{-\infty}^{\infty} v(\xi)\, S(\xi,x)d\xi = K(Sv)(x)$ where $S(\xi,x)$ is a symmetric kernel which does not however satisfy the requirements of the Fredholm theory, but it can be shown that the operator S is a bounded positive operator, i.e. that

$$0 < \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v(\xi)v(x)S(\xi,x)d\xi dx < M_C \int_{-\infty}^{\infty} \{v(x)\}^2 dx$$

for some constant M_C and all $v(x)$ for which $\int v^2 dx$ converges. It follows from the theory of bounded symmetric operators that (for fluid of infinite depth) the operator S has a continuous spectrum when $0 < K^{-1} < k^{-1}$ and a discrete spectrum when $k^{-1} < K^{-1} < M_C$ and a corresponding spectral decomposition, i.e. a Fourier type expansion of an arbitrary function as the sum of an integral and of a finite series. It also follows that the eigenvalues corresponding to the series can be found by the familiar variational principles, in particular

$$\frac{1}{K_1} = \max_u \frac{\iint u(\xi)u(x)S(\xi,x)d\xi dx}{\int u^2\, dx} .$$

It only remains to show that $(K_1)^{-1} > k^{-1}$, and this is done by Kelvin's minimum energy theorem: if the curve C_1 lies inside the curve C_2 then

$$\iint u(\xi)u(x)\ S_1(\xi,x)d\xi dx < \iint u(\xi)u(x)\ S_2(\xi,x)d\xi dx.$$

The existence of a trapping mode for an arbitrary curve C now follows from the existence of a trapping mode for a small circle. This argument, like Jones's, can be extended to finite constant depth and to humps on the bottom and it is much simpler than Jones's. It can be shown that for any arbitrary submerged curve C the number of trapping modes is finite. Arguments can be given to show that for suitable curves C symmetrical about a vertical line there are odd as well as even modes. These conclusions are the same as Aranha's. Our method is probably not very suitable for numerical work but has theoretical advantages.

References.

Aranha, J.A.P. 1986 Trapped wave and non-linear resonance in semi-submersible. Proc. First International Workshop on Water Waves and Floating Bodies.

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