Remarks on the numerical treatment of the intersection point between a rigid body and a free surface

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The direct numerical simulation of unsteady two-dimensional free surface flows using a boundary integral method and a Lagrangian time-stepping procedure has received considerable attention since the pioneering work of Longuet-Higgins and Cokelet /1/. One of the main difficulties encountered when a free surface piercing body is present is the proper description of the flow in the vicinity of the intersection between the rigid body and the free surface. Lin /2/ was the first to propose a numerical treatment which permitted to overcome this difficulty. However, Lin's method is empirical and a theoretical approach seems necessary in order to assess the domain of validity of such a treatment.

A possible method to deal with this problem in a somehow formal way is to first gain insight into the flow structure through an asymptotic study. Ideally, the proper behavior near the intersection point could then be accounted for in the numerical scheme. It is this approach which will be discussed here.

Asymptotic behavior (weakly nonlinear regime).

As in /2/, we consider a rectangular tank with a wavemaker at one end. Assuming a "sufficiently large" depth, an asymptotic study of the problem indicates that their exist two regimes for the flow: a *weakly nonlinear regime* for an acceleration of the wavemaker much smaller than that of gravity and an *impulsive regime* for an acceleration of the wavemaker much greater than that of gravity. The difficulties associated with the impulsive regime were discussed in /3/. Here, we restrict ourselves to the weakly nonlinear regime.

Non-dimensional variables are defined using the depth of the tank, H, as length scale and the acceleration of gravity, g, as acceleration scale. The wavemaker is assumed to be vertical (piston type). The behavior of the solution near the intersection point between the wavemaker and the free surface is then studied *a priori* using a method similar to that of Kravtchenko /4/. The resulting expansion of the complex velocity potential Ψ in the vicinity of the intersection point is (the origin of the complex plane is taken at the intersection point):

$$\Psi = (\lambda_0 + i\mu_0) + (\frac{d\ell}{dt} + i\frac{d^2\lambda_0}{dt^2})\, z + \frac{1}{\pi}\frac{d^3\ell}{dt^3}\, z^2 \log z + (\lambda_3 + \frac{i}{2}\frac{d^3\ell}{dt^3})\, z^2 + o(\|z^2\|), \tag{1}$$

where ℓ is the displacement of the wavemaker and λ_0, μ_0 and λ_3 are real-valued functions of time (which are not known a priori). This behavior is consistent — in the weakly nonlinear regime — with that derived from an expansion of the solution in a particular case by Roberts /5/.

Of interest to us in order to use a boundary integral method are the values of the velocity potential, Φ, and its normal derivative, Φ_n, along the boundary of the fluid domain.

On the x-axis, (1) yields:

$$\Phi = \lambda_0 + \frac{d\ell}{dt}\,(s_c - s) + o(|s - s_c|) \tag{2a}$$

$$\Phi_n = -\frac{d^2\lambda_0}{dt^2} - \frac{d^3\ell}{dt^3}\,(s_c - s) + o(|s - s_c|), \tag{2b}$$

where s is a curvilinear abscissa along the boundary and s_c the abscissa of the intersection point.

Similarly, on the y-axis, (1) yields:

$$\Phi = \lambda_0 - \frac{d^2\lambda_0}{dt^2}\,(s_c - s) + o(|s - s_c|) \tag{3a}$$

$$\Phi_n = -\frac{d\ell}{dt} + o(|s - s_c|). \tag{3b}$$

These expressions indicate that:

• Φ and Φ_n are continuously differentiable functions of s along the boundary, except at the intersection point s_c;

• Φ is continuous at the intersection point, but, in general, its tangential and normal derivative Φ_s and Φ_n are not.

We, therefore, state the following regularity hypotheses:

the flow is said to be weakly regular at the intersection if the potential and its normal derivative are continuously differentiable functions of the curvilinear abscissa along the boundary of the fluid, except, may be, at the intersection point where Φ_s and Φ_n can experience a finite jump.

The result of our asymptotic study is that the flow is weakly regular at the intersections in the weakly nonlinear regime. It can be shown that this result still applies for a non-vertical wavemaker (angle of intersection $\theta \in [0, \pi]$) [1].

Numerical treatment (nonlinear problem).

Numerical treatments at the intersection point have been derived from the preceding asymptotic study and implemented at IFP in the code SINDBAD (SImulation Numérique D'un BAssin De houle). A mixed Eulerian-Lagrangian method similar to /1/ is applied to solve the full nonlinear equations. Since similar methods have been widely discussed, we only insist here on the numerical treatment at the intersection. At each time-step, we need to solve for the harmonic function Φ knowing Φ along the free surface, Γ_d, and Φ_n along rigid boundaries, Γ_n. We start from the integral equation:

$$-\alpha(P)\,\Phi(P) + \int_{\Gamma_d+\Gamma_n} \Phi(Q)\,G_n(P,Q)\,ds_Q = \int_{\Gamma_d+\Gamma_n} \Phi_n(Q)\,G(P,Q)\,ds_Q, \tag{4}$$

where G is the Green function and α the local angle of the boundary. The boundary of the domain is approximated by segments and equation (4) is discretized assuming that Φ and Φ_n are linear along each segment [2]. In order to clearly distinguish nodes belonging to Γ_d from those belonging Γ_n, nodes are not located at the intersections [3].

A few different numerical treatments of the intersection points have been implemented. We discuss here three of them for the intersection between a vertical wavemaker and the free surface [4]. We call C the point located at the left intersection, P_1 the neighboring point on the wavemaker and P_M the neighboring point on the free surface — figure 1.

We assume respectively:

[0] a linear variation of Φ and Φ_n along the segment $P_M P_1$;

[1] a linear variation of Φ and Φ_n along the segments $P_M C$ and CP_1, given by (2) and (3). The values of λ_0, $\frac{d^2\lambda_0}{dt^2}$ and $\frac{d\ell}{dt}$ are expressed, using (2) and (3), in terms of the values of Φ and Φ_n at P_1 and P_M. $\frac{d^3\ell}{dt^3}$ is obtained directly from the wavemaker boundary condition;

[2] a linear variation of Φ and Φ_n along the segments $P_M C$ and CP_1, extrapolated from the variation along the segments $P_{M1} P_M$ and $P_1 P_2$, respectively.

Treatment [0] is a "no treatment"; it assumes that the flow is (strongly) regular at the intersection. Treatment [1] uses explicitly the local behavior (1). Treatment [2] assumes that the flow is weakly regular at the intersection.

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[1] The singularity is then in $z^{\frac{\pi}{\theta}}$ for the complex potential (if $\frac{\pi}{\theta}$ is not an integer). However, the strength of the singularity (i.e., its coefficient in the expansion of Ψ) is not, in this case, a local function of the boundary conditions.

[2] High-order polynomial expansions have not been used; they would not be consistent with the expansion (1).

[3] This ambiguity in the nature of nodes located at the intersections is a problem encountered, for instance, in /2/. Lin overcame this difficulty by writing that the intersection points belong to both boundaries. This implies a continuity hypothesis at the intersections which we want to make explicit here.

[4] Similar treatments are also applied at solid-solid corner points. Treatments [0] and [2] also apply to a non vertical wavemaker.

A first test has been performed by solving the *linear* problem for a displacement of the wavemaker (at rest for $t \leq 0$):

$$\ell = -\frac{a}{2}\cos(\omega t), \quad (t \geq 0) \tag{5}$$

As in /2/, (5) will be referred to as a sine wavemaker motion. The length of the tank is equal to 5 and the pulsation ω to $\frac{\pi}{2}$. The numerical solution as been compared at $t = 5$ to a quasi-analytic linear solution (see /6/). With 100 points on the free surface and a time step equal to 0.1, the relative error (L^2 norm / a) is approximatively equal to 0.02 when [0] is used and 0.005 when [1] or [2] are used. This result indicates that, for the linear problem:

• if one avoids to locate nodes at the intersection points, the weak singularity appearing in the linearized problem is accomodated without any special treatment, i.e., the numerical scheme does not blow-up and the accuracy is reasonable. Note, however, that some assumptions are implicitly made in this "absence" of treatment;

• taking into account properly the singularity appearing at the intersections does improve significantly the accuracy of the numerical procedure, at no additional computational cost;

• for a sufficiently fine grid, the same degree of accuracy is achieved either by taking into account explicitly the local behavior of the linear solution or by assuming the flow to be weakly regular at the intersection.

The treatment [2] of the intersection point has then been used to solve the fully nonlinear problem. A comparison with an analytic solution is not possible in this case. However, a comparison has been made with a numerical second-order solution, leading to a good agreement for the free surface profiles for a moderate acceleration of the wavemaker, /6/. Even for larger accelerations (breaking waves in the tank), the proposed treatment appears to be efficient. We show on figure 2 a comparison between SINDBAD and results from Lin /2/, the same discretisation being used (both in space and time). Both results agree fairly well, but a smoother behavior is obtained near the intersection.

Conclusions.

In order to deal with the problem of the numerical determination of the flow in the vicinity of a free surface piercing body, it is suggested that one should ensure that the numerical scheme is consistent with the behavior of the solution at the intersection. This method permits to make explicit the assumptions which are implicitly made in any numerical treatment of the intersection point. Good results have been obtained with treatment [2] which do not use explicitly the local behavior of the linear solution but only assumes the flow to be weakly regular at the intersections. These regularity hypotheses are similar to those made implicitly by Lin /2/ in his treatment. They are fully consistent with the local behavior of the solution in the weakly nonlinear regime.

However, more work remains to be done in order to get a better understanding of the impulsive regime and to derive an appropriate treatment of the intersection point in this case. Clearly, a first problem is to know if, on the length scale of the grid, the regularity hypotheses made here are consistent with the local behavior of the flow in this regime.

References.

/1/ Longuet-Higgins and Cokelet, 1976, "The Deformation of Steep Surface Waves on Water. I. A Numerical Method of Computation," *Proc. R. Soc. Lond.*, Vol. A 360, pp. 1-26.

/2/ Lin, 1984, "Nonlinear Motion of the Free Surface near a Moving Body," Ph.D. Thesis, MIT.

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/4/ Kravtchenko, 1954, "Remarques sur le Calcul des Amplitudes de la Houle Linéaire Engendrée par un Batteur," *Proceedings of the Fifth Conference on Coastal Engineering,* pp. 50-61.

/5/ Roberts, 1987, "Transient Free-Surface Flows Generated by a Moving Vertical Plate," *Q. J. Mech. Appl. Math.,* Vol. 40, Pt. 1.

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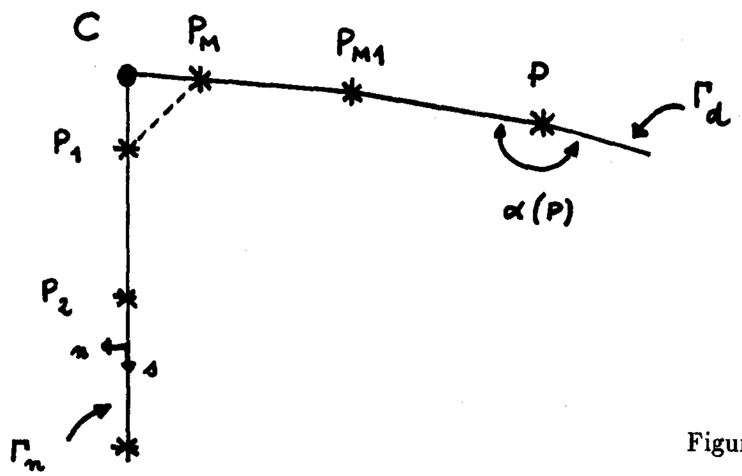


Figure 1 — Geometric definitions at the intersection

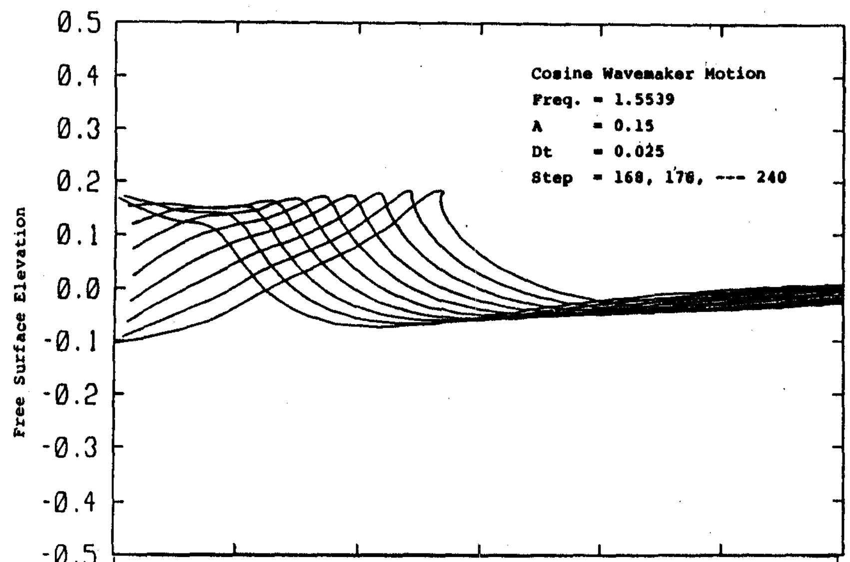
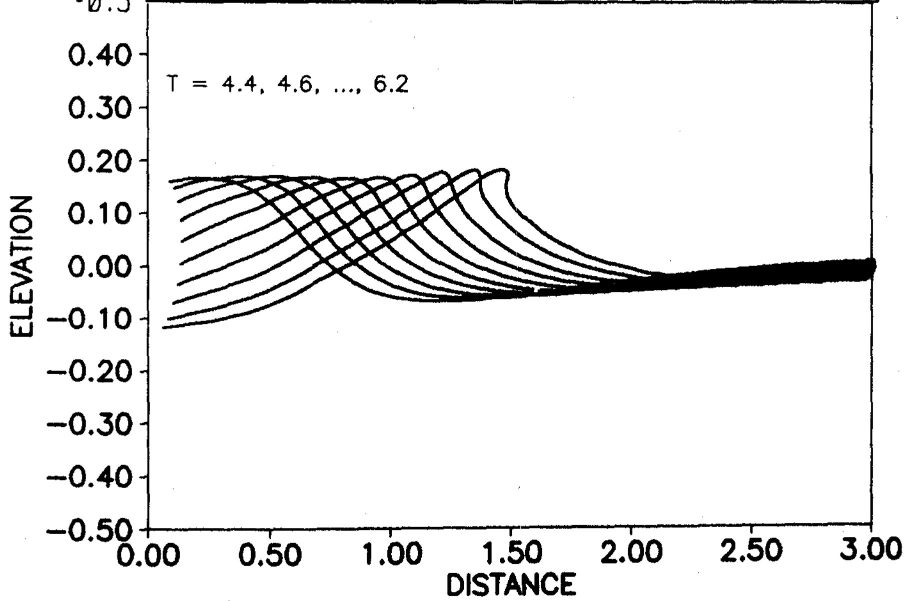


Figure 2 — Results from Lin /2/



— Results from SINDBAD

Yeung: In the Bristol Workshop, we reported that the intersection-point treatment of Lin *et al.* (1984) suffers a convergence problem as the grid size is reduced. Using a quasi-steady analysis, we showed (Wu & Yeung, 1987) that the potential behaves locally like $z \log z$ for a 90 degree intersection because of the confluence of the boundary conditions. Effects of intersection angle on the singularity were also discussed at that time.

I am pleased that more work on this problem has been carried out by the author. The present analysis, together with A.J. Roberts, suggests that the linearized free-surface condition has reduced the singular behavior to a weaker type, $z^2 \log z$. This is obviously more amenable to numerical treatment, but it is subject to the restriction that body acceleration is small compared to gravitational acceleration. Since nonlinear fluid motion of interest is often the result of $O(g)$ acceleration, is it not true that the proposed extrapolator treatment will suppress the genuine fluid motion near the wall? In fact, the extrapolator scheme proposed has the peculiar property that the fluid motion does *not* depend on the acceleration of the wall at all!

Ref: Wu, C. F. & Yeung, R. W. "Nonlinear Wave-Body Motion in a Closed Domain", 2nd Int. Workshop on Water Waves & Floating Bodies, Bristol, March, 1987

Cointe: I thank you for your comments. In your Bristol paper, you studied the problem after discretization in time (which is a Neuman-Dirichlet boundary value problem). You showed neatly how the confluence of boundary conditions yields, in this case, a singularity at the intersection, in $z \log z$ for the complex potential at a $\frac{\pi}{2}$ corner. This result was in agreement with the local expansions of the solution of the impulsive problem that were carried out by Lin both for the small time expansion ($\Phi = 0$ for $y = 0$) and the linear expansion ($\Phi_{tt} + \Phi_y k = 0$ for $y = 0$).

However, these results puzzled me because, as early as 1954, Kravtchenko found a $z^2 \log z$ singularity for the linear harmonic problem ($-\omega^2 \Phi + \Phi_y = 0$ for $y = 0$). Actually, Roberts showed that Lin's expansion of the linear problem is incorrect. He found for the linear solution of the impulsive problem a rather strange behavior at the intersection (strongly oscillatory). However, Roberts showed that the solution for a non-impulsive problem is smoother, and that linear theory is only self-consistent for a bounded acceleration of the wavemaker. In this case, it is possible to check from Robert's expansion that the $z^2 \log z$ behavior is recovered, in agreement with Kravtchenko's old result.

A rather simple explanation for these results has been proposed: the equations should only be linearized in the weakly linear regime, *i.e.*,for $|\Gamma| << g$; in this case the $z^2 \log z$ behavior for the transient problem can be exhibited *a priori*. The singularity of the problem after discretization in time is, therefore, misleading. The difficulty is obviously to be able to account for the real singularity in the numerics. In the weakly nonlinear regime, this has been achieved here either by using explicitly the local behavior of the linear solution (treatment [1]) or by simply making regularity hypotheses consistent with the linear solution (treatment [2]). I do agree with you that the treatment [1] is *a priori* subject to the restriction that the acceleration is much smaller than that of gravity. This is not the case, however, for treatment [2] which only assumes the flow to be "weakly regular," which is a regularity assumption similar to that made implicitly by Lin. I therefore expect treatment [2] to be accurate as long as it ensures the convergence of the numerical scheme. Presumably, only computations can show an upper limit for the acceleration of the wavemaker. This limit could be quite large because, even in the impulsive regime, I do not believe the local behavior of the flow to be $z \log z$. Locally, the nonlinear terms in the free-surface boundary condition should be retained, so that the $\Phi = 0$ boundary condition is not appropriate. Obviously, more work needs to be done along this direction.

Schultz: You indicate, as we have, that $d\eta/dx = -\alpha$ at the wall, where α is the acceleration of the wall. Have you checked if this condition has been met by your boundary integral calculations? It would not be surprising if this were not the case, since boundary integral methods are not accurate at corners, even if there is no singularity.

Cointe: For the linear solution of the problem, expansion (1) does indeed imply $d\eta/dx = -\alpha$ where α is equal to Γ/g. I thank you for suggesting to check if this condition is satisfied in the numerical computation. I used SINDBAD (with treatment [2]) to compute the linear solution corresponding to a sine wavemaker motion for which the velocity is continuous at $t = 0$. The results are shown on the figure: a good overall agreement is obtained, but oscillations occur at the beginning of the computation. This is very likely due to the fact that the acceleration experiences a jump at $t = 0$.