A HIGH-ORDER EQUATION FOR SHALLOW-WATER WAVES

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Summary

In an earlier work on a spectral method for nonlinear wave-wave and wave-body interactions (Dommermuth & Yue, 1987, hereafter denoted as DY), we presented in passing and without derivation a closed-form equation for the case of shallow-water waves, which was in error. In this note we rederive the set of shallow-water evolution equations which is generalized to an arbitrary high order in the shallowness parameter. The equations show that there is an upper bound on the wavenumber, and hence resolution, of this approximation. As a demonstration of their usefulness, high-order forms of the equations are integrated for the case of long wave generation by a moving surface pressure, and the results are compared to the computational and experimental data of Wu & Wu (1982), Ertekin (1984) and others.

Formulation

Consider the irrotational motion of a homogeneous, incompressible and inviscid fluid under a free surface in depth h. A velocity potential $\Phi(\mathbf{x},z,t)$ ($\mathbf{x}=(x,y)$ is in the mean horizontal plane and z is positive upward) exists which is harmonic within the fluid. If we define the surface potential:

$$\Phi^S(\mathbf{x},t) = \Phi(\mathbf{x},\eta(\mathbf{x},t),t) \tag{1}$$

where $z = \eta(\mathbf{x},t)$ is the free surface which is assumed to be continuous and single-valued, the kinematic and dynamic boundary conditions on the free surface can be written in the form:

$$\eta_t + \nabla_\mathbf{x}\Phi^S\cdot\nabla_\mathbf{x}\eta - (1+\nabla_\mathbf{x}\eta\cdot\nabla_\mathbf{x}\eta)\ \Phi_z(\mathbf{x},\eta,t) = 0 \tag{2a}$$

$$\Phi^S_t + \eta + \frac{1}{2}\nabla_\mathbf{x}\Phi^S\cdot\nabla_\mathbf{x}\Phi^S - \frac{1}{2}(1+\nabla_\mathbf{x}\eta\cdot\nabla_\mathbf{x}\eta)\ \Phi^2_z(\mathbf{x},\eta,t) = -P_a \tag{2b}$$

where $\nabla_\mathbf{x} \equiv (\partial/\partial x,\partial/\partial y)$ denotes the horizontal gradient and $P_a(\mathbf{x},t)$ is the

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prescribed surface pressure. (For simplicity, we choose time and mass units so that gravitational acceleration and fluid density are unity, and all spatial coordinates are normalized by typical wavelength.)

For shallow-water waves, $h \ll 1$, we expand $\Phi(x,z,t)$ in a power series in z about the bottom $z=-h$, so that we have in general (e.g. Mei 1983):

$$\Phi(x,z,t) = \phi - \frac{(z+h)^2}{2!}\nabla_x^2\phi + \frac{(z+h)^4}{4!}\nabla_x^4\phi - \frac{(z+h)^6}{6!}\nabla_x^6\phi + \ldots$$

$$\equiv \mathrm{COS}[(z+h)\nabla_x]\,\phi \tag{3}$$

where $\phi(x,t)$ is the two-dimensional potential of Φ evaluated on the bottom, and the COS operator is defined as:

$$\mathrm{COS}[(z+h)\nabla_x] \equiv \sum_{n=0}^{\infty} \frac{(-1)^n (z+h)^{2n}\,\nabla_x^{2n}}{(2n)!} \tag{4}$$

The vertical particle velocity is then:

$$\Phi_z(x,z,t) = \{ -\frac{(z+h)}{1!}\nabla_x^2 + \frac{(z+h)^3}{3!}\nabla_x^4 - \frac{(z+h)^5}{5!}\nabla_x^6 + \ldots \}\,\phi(x,z,t)$$

$$\equiv -\mathrm{SIN}[(z+h)\nabla_x] \bullet \nabla_x\phi \tag{5a}$$

where

$$\mathrm{SIN}[(z+h)\nabla_x]\bullet\nabla_x \equiv \sum_{n=0}^{\infty} \frac{(-1)^n (z+h)^{2n+1}\,\nabla_x^{2n+2}}{(2n+1)!} \tag{5b}$$

Following DY, we express the surface vertical velocity, $\Phi_z|_{z=\eta}$, in terms of ϕ, which in turn is solved in terms of the surface potential, Φ^S. Thus, we write on the free surface, $z=\eta$:

$$\Phi_z(x,\eta,t) = -\mathrm{SIN}[(\eta+h)\nabla_x] \bullet \nabla_x\phi \tag{6}$$

and from (2):

$$\Phi^S(x,t) \equiv \Phi(x,\eta,t) = \mathrm{COS}[(\eta+h)\nabla_x]\,\phi$$

$$= [1 - f_2 + f_4 - f_6 + \ldots]\,\phi(x,t) \tag{7}$$

where $f_n \equiv \frac{1}{n!}(\eta+h)^n\,\nabla_x^n$

Clearly, f_n's are in general not commutative, and also, $f_{2n} \neq f_n f_n \equiv f_n^2$, for example. The problem is complete if we formally invert (7) and write:

$$\phi(x,t) = \mathrm{SEC}[(\eta+h)\nabla_x]\ \Phi^S(x,t) \qquad \text{for } |(\eta+h)\nabla_x| < \frac{\pi}{2} \tag{8}$$

so that

$$\Phi_z(x,\eta,t) = -\mathrm{SIN}[(\eta+h)\nabla_x] \bullet \nabla_x\ \mathrm{SEC}[(\eta+h)\nabla_x]\ \Phi^S(x,t) \tag{9}$$

Whence, we finally obtain closed-form expressions of the free-surface evolution equations:

$$\eta_t + \nabla_x\Phi^S\bullet\nabla_x\eta + (1+\nabla_x\eta\bullet\nabla_x\eta)\ \{\mathrm{SIN}[(\eta+h)\nabla_x]\bullet\nabla_x\mathrm{SEC}[(\eta+h)\nabla_x]\ \Phi^S\} = 0$$

$$\Phi_t^S + \eta + \frac{1}{2}\nabla_x\Phi^S\bullet\nabla_x\Phi^S - \frac{1}{2}\ (1+\nabla_x\eta\bullet\nabla_x\eta)\ \{\mathrm{SIN}[(\eta+h)\nabla_x]\bullet\nabla_x\mathrm{SEC}[(\eta+h)\nabla_x]\ \Phi^S\}^2 = -P_a \tag{10}$$

In general, the SEC operator, which is the inverse of the COS operator, can be written out, by inspection, to any order of approximation. We give here only the first six terms:

$$\begin{aligned}\mathrm{SEC}[(\eta+h)\nabla_x] &= 1 + f_2 + [f_2^2-f_4] + [f_2^3-(f_2f_4+f_4f_2)+f_6] \\ &+ [f_2^4-(f_2^2f_4+f_2f_4f_2+f_4f_2^2)+(f_2f_6+f_4f_4+f_6f_2)-f_8] \\ &+ [f_2^5-(f_2^3f_4+f_2^2f_4f_2+f_2f_4f_2^2+f_4f_2^3)+(f_2^2f_6+f_2f_4^2+ f_2f_6f_2+f_4f_2f_4+f_4^2f_2+f_6f_2^2) \\ &\qquad -(f_2f_8+f_4f_6+f_6f_4+f_8f_2)+f_{10}] \\ &+ O[(\eta+h)\nabla_x]^{12}\end{aligned} \tag{11}$$

In the special case of constant η+h, the spatial operators are commutative, (11) reduces to the form of the ordinary Taylor expansion of secant, and (10) reduces to Eqs.(2.10) of DY.

In order for the perturbation series to be convergent, there is an upper bound on the resolution of the present approximation, given by $|(\eta+h)\nabla_x| < \pi/2$, which is the radius of convergence of (8). For example, if we represent the surface potential, Φ^S, in Fourier series, the above inequality places a specific limit on the maximum horizontal wavenumber, relative to (η+h), that can be used.

Numerical Method & Results

Given initial conditions $\Phi^S(x,0)$ and $\eta(x,0)$, prescribed forcing $P_a(x,t)$, and suitable boundary conditions, (10) can be integrated directly in time for any specified order of approximation, M. For large M, (10) requires high-order spatial derivatives of Φ^S and η, so that for computations, it is useful to represent Φ^S and η as spectral series, so that the spatial derivatives are calculated analytically. Thus, depending on boundary conditions in x, orthogonal basis functions such as Fourier, Fourier-Bessel, Chebyshev, Legendre etc. may be employed. If explicit time integrators (such as Runge-Kutta) are used for (10), matrix inversion is not required, and the computational effort is dictated only by the necessary projections between the physical and spectral spaces. With the use of fast transforms, the operation count is typically linearly proportional to the number of spectral modes, N, and order M. As with the non-shallow-water case of DY, exponential convergence with respect to both M and N can be expected, subject to the validity and convergence of (3). This is confirmed by our numerical experiments.

For illustration, we apply the present theory to the periodic generation of upstream solitons by a translating surface pressure disturbance (e.g. Wu & Wu, 1982, Ertekin, 1984). Specifically, different high-order forms of the evolution equations (10) are integrated and the results compared to available computational and experimental data.

References

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Ertekin, R.C. (1984). Soliton generation by moving disturbances in shallow water: theory, computation and experiments. Ph.D. dissertation, U. Calif., Berkeley.

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Wu, D.M. & Wu, T.Y. (1982). Three-dimensional nonlinear long waves due to moving surface pressure. Proc. 14th Symp. Naval Hydrodyn., National Academy Press, Washington, D.C., 103-125.

Chin: The authors make use of pseudo-differential operator theory to derive high-order shallow water equations. The condition $|(\eta + h)\nabla_x| < \pi/2$ makes sense in that setting. Physically, this must be interpreted as a ratio of wave height to wave length. In this sense it is consistent with the concept of long wave theory. Here, we must also require that $max\,|\eta| \leq h << 1$.

The numerical computations at large time seem to be contaminated by numerical dispersion at least at the leading and trailing edges of the primary pulse.

Dommermuth & Yue: The high-wavenumber oscillations we observe are probably non-physical and related to the lack of smoothness of the cosine pressure distribution used. The wavelengths of these oscillations are somewhat greater than grid spacing and are not likely to be a result of grid-scale instabilities.

Schultz: You state that in previous calculations you filtered your pseudo-spectral results. Do you do this by truncation of unaliased or aliased coefficients?

Dommermuth & Yue: For the present set of calculations, no filtering or smoothing of high-wavenumber modes was done, although in view of some grid-scale error growth, such filtering may indeed be desirable. In our previous work (Dommermuth & Yue, 1987), we have found it useful to apply an ideal (or smoothing) filter to remove a small fraction of the highest *unaliased* modes near the wavenumber truncation for some applications.

Cooker: You have a condition $|(\eta+h)\nabla_x\phi| < \frac{\pi}{2}\phi$. If you add a constant K value to ϕ everywhere on the free surface, the solution cannot be changed (only $\nabla\phi$ is a physical quantity). By making K arbitrarily large, the right-hand side of this condition can be made arbitrarily large, thus ensuring its validity. Can you clarify this point?

Dommermuth & Yue: For shallow water, we write an expansion about the bottom, $z = -h$, so that we have even powers of height $(\eta+h)$. In this sense, h is not arbitrary but a parameter of the problem.

Wehausen:

1. I am confused by the inequality $|(\eta+h)\nabla_x| < \frac{\pi}{2}$, which appears to be important. Can you clarify what is meant by it?

2. Your trailing "solution" is moving to the left in the coordinate system fixed on the pressure distribution. Is it really moving to the right, but with subcritical velocity? Do you have any experimental evidence for its occurrence?

Dommermuth & Yue:

1. Perhaps $|(\eta+h)\nabla_x\phi| < \frac{\pi}{2}\phi$, or in fact $k(\eta+h) < \frac{\pi}{2}$, where k is the maximum wavenumber represented, is clearer.

2. The solitary-wave-like disturbance behind the moving pressure (which has critical speed) is moving forward (with the pressure) but with a subcritical speed. The disturbance has a steeper rear slope. We have not made any physical observation of this problem.

Evans: I believe the same operator technique was used by Miles in a paper in *JFM* in 1985 in which he was concerned with fundamental sloshing frequencies in shallow closed basins. He in turn quotes Sen (1927) as the first person to use the technique.

Dommermuth & Yue: We thank Professor Evans for pointing out the references to Miles (1985) and Sen (1927) who introduced cosh and sinh operators (defined by their power-series) for linear waves over a variable bottom. Our technique is similar in idea but not related to their method.