Edge waves over a sloping beach

D.V. Evans
Department of Mathematics

Introduction

The existence of edge waves over a sloping beach has been known since 1846 when Stokes(1), on the basis of linear water-wave theory, produced a simple solution of the governing equations and boundary conditions, which described a wave which was bounded in amplitude at the shoreline, and which decayed in a direction out to sea.

Over a hundred years after Stokes' solution Ursell(2) showed that it was just the first of a finite number of bounded edge wave modes - the precise number depending on the beach slope, and produced solutions in the form of a finite sum of exponential satisfying the linearised equations and boundary conditions. No explanation was given as to how the solution had been arrived at. Roseau(3), apparently unaware of Ursell's work, developed a systematic approach to constructing edge wave modes based on an integral representation which converted the problem into a functional equation. He derived bounded edge-wave solutions which can be transformed into the Ursell solutions as well as solutions having singularities at the shoreline.

Whitham(4) showed how the Ursell edge wave modes could be determined systematically using an approach similar to, but simpler than, that of Roseau.

Here we show how Whitham's method can be used to construct edge wave modes over a sloping beach on which a mixed boundary condition is satisfied. Such a problem arises when certain effects of rotation are included, or again, in the case of a stratified fluid having an exponentially varying density over a sloping beach. This later problem has been considered by Greenspan(5) who has written down edge-wave solutions. The solution obtained here agrees with the Greenspan solution and reduces to the Ursell solution as a special case.

Formulation

Cartesian axes are chosen with $y = 0$ $(x > 0)$ the horizontal plane of the undisturbed water surface, the z-axis the shoreline and y measured vertically upwards. The fluid occupies $y + x \tan \beta > 0$, $x > 0$, $\beta \leq \frac{\pi}{2}$, $-\infty < z < \infty$. On linear water-wave theory there exists a velocity potential $\Phi(x,y,z,t)$. We seek solutions in the fluid region which are simple harmonic in time, wave-like in the (long-shore)z-direction,and which vanish as $x \to \infty$ (out to sea).

Thus we assume

$$\Phi(x,y,z,t) = \text{Re}\ \phi e^{\pm ikz} e^{-i\omega t},\ k > 0 \tag{2.1}$$

whence $\phi(x,y)$ satisfies

$$(\nabla^2 - k^2)\phi = 0,\ y + x \tan\beta > 0,\ x > 0 \tag{2.2}$$

$$\lambda\phi - \phi y = 0,\ x > 0,\ \lambda = \omega^2/g \tag{2.3}$$

$$\phi \to 0,\ y + \tan\beta > 0,\ x \to \infty \tag{2.4}$$

Finally
$$\phi_n + \alpha\phi = 0, \qquad \text{on} \quad y + x\tan\beta = 0 \tag{2.5}$$

where n is the normal into the fluid and α is a real constant.

We seek non-trivial solutions to (2.2) - (2.5), for k, α β given, anticipating that this will only be possible for certain $\lambda(= \omega^2/g)$. We note first that a simple Stokes-type solution exists, namely

$$\phi = \exp\{-\ kx\cos\mu + ky\sin\mu\}. \tag{2.6}$$

This satisfies (2.2), (2.4) for $0 < \mu < \frac{\pi}{2}$, and (2.3) provided = $k\sin\mu$. It also satisfies (2.5), which we write as

$$\phi_y\cos\beta + \phi_x\sin\beta + \alpha\phi = 0, \qquad \text{on} \quad y\cos\beta + x\sin\beta = 0$$

provided $\alpha = k\sin\chi$ and $\mu = \beta - \chi$.

Thus given β, α, k with $\alpha < k$, then

$$\lambda = k\sin(\beta - \chi)$$

where
$$\alpha = k\sin\chi \quad \text{defines } \chi, \quad \text{so that}$$

$$\lambda = (k^2 - \alpha^2)^{\frac{1}{2}}\sin\beta - \alpha\cos\beta$$

We seek to generalise this result to produce further bounded solutions for $\alpha > 0$ similar to the Ursell solutions for $\alpha = 0$, namely,

$$\begin{aligned}\phi(x,y) &= \exp\{-\ kx\cos\beta + ky\sin\beta\} \\ &+ \sum_{m=1}^{n} A_{mn}[\exp\{-\ k[x\cos(2m-1)\beta + y\sin(2m-1)\beta]\} \\ &+ \exp\{-\ k[x\cos(2m+1)\beta - y\sin(2m+1)\beta]\}\end{aligned} \tag{2.7}$$

where

$$A_{mn} = (-1)^m \prod_{r=1}^{m} \frac{\tan(n-r+1)\alpha}{\tan(n+r)}$$

and
$$\lambda = k\sin\mu \quad \text{where} \quad \mu = (2n+1)\beta$$

3. Solution

Equation 2.2 is satisfied by exponentials of the form

$$\exp\{\tfrac{k}{2}(\zeta + \zeta^{-1})x \pm i\tfrac{k}{2}(\zeta - \zeta^{-1})y\}$$

for any ζ. We therefore seek a solution in the form

$$\phi(x,y) = \frac{1}{2\pi i}\int_C \{f(\zeta)\exp\tfrac{k}{2}\{(\zeta+\zeta^{-1})x+i(\zeta-\zeta^{-1})y\}+g(\zeta)\exp\tfrac{k}{2}\{(\zeta+\zeta^{-1})x-i(\zeta-\zeta^{-1})y\}\}d\zeta \quad (3.1)$$

for some f, g, C.

Application of the free surface condition (2.3) shows that

$$(\zeta+\ell)(\zeta-\bar{\ell})f(\zeta) = (\zeta+\bar{\ell})(\zeta-\ell)g(\zeta) \quad (3.2)$$

whilst it can be shown that the conditions (2.4) (2.5) are satisfied if

$$\lambda = k \sin\{(2n+1)\beta-\chi\}$$

where

$$\alpha = k \sin \chi$$

and

$$f(\zeta) = \frac{1}{\zeta}\frac{\zeta-\ell}{\zeta+\ell}\prod_{s=1}^{n}\frac{(\zeta-\ell\omega^{-s})(\zeta-\bar{\ell}\omega^{s})}{(\zeta+\ell\omega^{-s})(\zeta+\bar{\ell}\omega^{s})}$$

Here $\ell = e^{i\mu}$, $\lambda = k \sin \mu$, and $\omega = e^{2i\beta}$.

The contour C encloses all the poles of $f(\zeta)$, $g(\zeta)$ except at $\zeta = 0$. The resulting contributions from the residues at the poles after multiplication by a normalising constant, produce the solution

$$\phi(x,y) = e^{-kx\cos(\beta-\chi) + ky\sin(\beta-\chi)}$$

$$+ \sum_{m=1}^{n} A_{mn}\Big\{e^{-kx\cos\{(2m-1)\beta-\chi\} - ky\sin\{2m-1)\beta-\chi\}}$$

$$+ C_{mn}e^{-kx\cos\{(2m+1)\beta-\chi\} + ky\sin\{(2m-1)\beta-\chi\}}\Big\} \quad (3.3)$$

where

$$C_{mn} = \tan\,(m\beta-\chi)/\tan\,m\beta \tag{3.4}$$

$$A_{mn} = \frac{(-1)^m \tan m\beta}{\tan(m\beta-\chi)} \prod_{r=1}^{m} \frac{\tan(n+1-r)\beta}{\tan r\beta} \frac{\tan(r\beta-\chi)}{\tan\{(r+n)\beta-\chi\}} \tag{3.5}$$

It can be seen that in agreement with Greenspan(5) The Ursell solution is obtained for $\alpha = 0$.

References

1. G.G. Stokes (1846) Report on recent researches in hydroynamics. Rep. 16th Brit Assoc.Adv.Sci. Southampton, Murray, London; 1-20.
2. F. Ursell (1952) Edge waves on a sloping beach. Proc.Roy.Soc.Lond. A, 214; 79-97.
3. M. Roseau (1958) Short waves parallel to the shore over a sloping beach. Comm. Pure Appl. Maths. 11, 433-493.
4. G.B. Whitham Lectures on Wave Propagation, New York, Springer. For Tata Institute of Fundamental Research 1979.
5. H.P. Greenspan, (1970) A note on edge waves in a stratified fluid. Stud.Appl.Math. 49; 381-388.

Martin: Can you prove that your 'generalized' edge waves do not exist when α is large? In other words, have you considered the other special case, $\alpha \to \infty$, when the boundary condition on the sea floor becomes $\phi = 0$.

Evans: For finite α we must have $\alpha < k$ so that the sets of poles and zeros, all lie on the unit circle in the method described here. However, I believe the $\alpha = \infty$, or $\phi = 0$ on the beach, case also permits edge wave solutions which are best derived by putting $\alpha = 0$ to begin with and arise as a simple example of Whitham's method. I would surmise that they are slightly more complicated than the Ursell edge wave modes, since the fundamental mode is itself probably a sum of two exponentials in contrast to the simple Stokes solution.

Ursell: Does your method also give the continuous spectrum (propagating waves)?

Evans: The method, which is due to Whitham, does indeed give the continuous spectrum although I have not worked it through for this mixed condition to see if it gives the same result as obtained, for example, by Peters who considers the same problem in 2-D, *i.e.*with $k = 0$. Whitham shows how the continuous spectrum can be derived in the simple problem of a rigid bed condition.

Kleinman:

1. Is there a relation between your solution and a Weiner-Hopf treatment and if they are not the same can you use your solution to go backwards and determine the appropriate factorization in a Weiner-Hopf approach?

2. Your problem looks a lot like the impedance wedge problem for the Helmholtz equation. Could you employ the Kantorovich-Lebedev transform to good advantage?

Evans:

1. I do not see how one can convert a problem in a wedge to one in a half-plane or strip which is the normal geometry for Weiner-Hopf problems. Conformal mapping is not available for the modified Helmholtz equation and any other type of transformation which converts the problem into one in a half-plane would surely leave the boundary conditions intractable.

2. I do not think Kantorovich-Lebedev transforms will work when we have both the Helmholtz equation *and* mixed condition on each face.