Trapping Modes Above Bodies in Finite Depth

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Ursell[1951] showed that the existence of trapping modes above an infinitely long submerged horizontal circular cylinder in deep water, depended upon the vanishing of a certain infinite determinant. He was also able to show that the determinant did indeed vanish provided that the radius of the cylinder was 'small enough' and that the wave-number $K(\equiv\sigma^2/g)$ was less than, but near to, k (the parameter in the modified Helmholtz equation).

The restriction on the radius of the cylinder was not due to any physical reasoning but purely to obtain a tangible result. Jones[1953] proved the existence of these modes over totally submerged cylinders, with no restriction on their size but symmetrical about a vertical plane, in infinite and finite depth using very deep arguments from functional analysis. Also, in that paper, Jones proved that trapping modes exist above protrusions (again symmetrical about a vertical plane) on the flat sea bed.

In a recent paper, Ursell[1987], attacked and solved the same problems as Jones, but using simpler arguments based upon Kelvin's minimum-energy principle of classical hydrodynamics. His proof relied on the knowledge that a mode for a small circular cylinder (deep water) had been shown to exist. Ursell's arguments were for infinite depth but require little modification for finite depth provided that a mode exists above a circular cylinder of small radius in finite depth.

Although this follows from Jones, it is possible to extend the method given by Ursell to finite depth. It is also possible to prove the existence of a trapping mode above a circular hump protruding from the sea-bed. These extensions will be described here.

Helmholtz multipoles are constructed which satisfy the free-surface condition and the boundary condition on the flat sea bed. These potentials are situated at a depth f below the free-surface and they are of unknown strengths. Summing over these potentials and applying the appropriate boundary condition on the cylinder, we find that the problem reduces to solving an infinite set of equations in an infinite number of unknowns (the multipole 'strengths') of the form

$$A_t + \sum_{n=1}^{\infty} A_n \psi_{nm}(K,k) \ , \ t=1,2,\ldots$$

where it can be shown that the (known) coefficients satisfy $\Sigma\Sigma|\psi_{nm}| < \infty$, provided that the cylinder is totally submerged and that it does not touch the sea-bed. Ursell has shown that this system of equations will have a non trivial solution if and only if its infinite determinant vanishes. It is noteworthy that the finite depth zero-flux condition imposes a more stringent inequality between K and k. It is found that, for the spectrum to be discrete, we need $K < k\tanh(kh)$ where h is the depth of the fluid.

If K is assumed to be near $k\tanh(kh)$ and that ka is small, it can be shown that the infinite determinant vanishes.The approximate dispersion relation for the cylinder is found to be of the form

$$\sigma^2/g \approx \{1-\tfrac{1}{2}\frac{(\beta\pi ka)^2}{D^2}\}\tanh[kh\{1-\tfrac{1}{2}\frac{(\beta\pi ka)^2}{D^2}\}]$$

where $\beta(kh,kf)$ and $D(kh)$ are complicated functions of their respective variables which tend to the following as h tends to infinity

$$\beta \longrightarrow 3\exp\{-2kf\}\ ,\ D \longrightarrow 1$$

agreeing with Ursell.

To construct the Helmholtz multipole potentials appropriate for the case of a circular hump, we can proceed direct from the potentials situated at (0,f). By subtracting out the 'image potential due to the sea-bed' and letting $f \longrightarrow h$, we derive the necessary potentials. It should be noted that the singular terms (modified Bessel functions of the second kind) of order 2n+1 vanish identically. As before, we now require that the hump does not come into contact with the free-surface for the convergence of the known coefficients.

In all three cases, the coefficients, and their respective determinants, are complicated, and nothing is known analytically about other roots for general values of the non-dimentionalised parameters. A numerical study by McIver & Evans[1985] in which the zeros of the determinant given by Ursell were sought will be extended for the other two determinants.

REFERENCES

Jones, D.S. (1953) The eigenvalues of $\nabla^2 u+\lambda u=0$ when the boundary conditions are given on semi-infinite domains. Proc. Camb. Phil. Soc., **49**, part 4, 668-684

McIver, P. and Evans, D.V. (1985) The trapping of surface waves above a submerged horizontal cylinder. *J. Fluid Mech.* **151**, 243-255

Ursell, F. (1951) Trapping modes in the theory of surface waves. *Proc. Camb. Phil. Soc.*, **47**, 347-358

Ursell, F. (1987) Mathematical aspects of trapping modes in the theory of surface waves. *J. Fluid Mech.* **183**, 421-437