AN EXPANSION OF THE SOURCE POTENTIAL AND ITS APPLICATIONS

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1 INTRODUCTION

For some analyses in infinite water depth, the availability of a series expansion for the source potential similar to that of John's representation in finite water depth is highly advantageous. For example, providing all the observational coordinates and the source coordinates can be separated, and the wave free modes decrease rapidly with the radial distances, this form of source potential will greatly facilitate the analytical evaluation of the troublesome free surface integral arising in the study of second order diffraction problems. This was the original motivation of the present work. An analytical decomposition of the infinite water depth source potential has been achieved and is given below. In particular, for source points lying on the free surface the proposed expression only involves some elementary functions and special functions and they may be readily evaluated. In view of this, it also suggests that the first order potential could be similarly expanded on the free surface whereever the expansion of the source potential is numerically efficient. This point will be further elaborated below. The present work is also concerned with the irregular frequencies. It will be shown numerically that now irregular frequencies can be removed by properly perturbing this expansion and coupling with the original source potential.

2 EXPANSION OF THE SOURCE POTENTIAL IN INFINITE WATER DEPTH

The work follows closely that of Martin[1], but some modifications have been made which considerably increase its domain of validity. By using the same notation and the definition of the source potential as given in Ref.1, it can be shown that

$$G_o(\rho,\alpha,y;\rho_o,\alpha_o,y_o) = \sum_{m=0}^{\infty} \frac{1}{2}\,\epsilon_m\, g_m(\rho,y;\rho_o,y_o)\cos m(\alpha-\alpha_o)$$

$$= \sum_{m=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{2}\,\epsilon_m\, \Lambda_m^j(\rho,y)\,\Psi_m^j(\rho_o,y_o)\cos m(\alpha-\alpha_o)$$

In particular, for $(\rho_o^2+y_o^2) > (\rho^2+y^2)$, one obtains

$$\Lambda_m^0(\rho,y) = -\pi K\, e^{-ky}\, J_m(K\rho)$$

$$\Psi_m^0(\rho_o,y_o) = e^{-Ky_o}[\,-2iH_m^{(1)}(K\rho_o) + (-1)\,S_{-m}^m(K\rho_o) - Y_m(K\rho_o)\,] - \frac{2P_m^m(\cos\theta_o)}{\pi(Kr_o)^{m+1}}$$

$$+\, e^{-Ky_o}\,[\,\frac{2}{\pi^{3/2}}\,\Gamma(m+\tfrac{1}{2})\,(\frac{2\rho_o}{K})^m \int_0^{y_o} dt\, \frac{e^{Kt}}{(\rho_o^2+t^2)^{m+1/2}}\,]$$

$$\Lambda^j_m(\rho,y) = \frac{2(2j)!}{k^{2j}} r^m \sum_{q=2j}^{\infty} \frac{(-Kr)^q}{(2m+q)!} P^m_{m+q}(\cos\theta) \qquad j>0$$

$$\Psi^j_m(\rho_o,y_o) = \frac{P^m_{m+2j}(\cos\theta_o)}{r_o^{m+2j+1}} + \frac{K}{2j} \frac{P^m_{m+2j-1}(\cos\theta_o)}{r_o^{m+2j}} . \qquad j>0$$

Here S_m is the Struve function of order m, and (ρ,α,y) and (r,θ,α) are cylindrical and spherical coordinates respectively. In order to verify the validity of the above alternative representation of G_o, some numerical results for the real part of g_m are illustrated in table 1. Comparisons have been obtained by performing a cosine transform of the source potential which is in turn calculated from 'FINGREEN'. In the evaluation of the infinite series in j, the summation is terminated when the relative differences between three successive values are less than a prescribed tolerance of 10^{-6}. Under this degree of convergence, it can be seen that the comparisons are very satisfactory.

3 APPLICATION I - EVALUATION OF THE FREE SURFACE INTEGRAL

It is well known that one of the crucial parts in the second order diffraction analysis is the accurate and efficient evaluation of the free surface integral. For this calculation in finite water depth, it has been found advantageous to use the procedure (see Ref. 2) of dividing the free surface into near field and far field regions. While numerical quadrature is employed for the near field integration, analytical evaluation is performed in the far field by first integrating explicitly in the azimuth angle. However, it can be shown that if the classical version of the source potentail is employed, the same methodology is rather expensive to apply in the case of infinite water depth. Unless a large domain for the near field integration is enclosed, neither the source nor the first order potential can be accurately expressed in a form which will facilitate the analytical integration in the far field. Consequently, the prohibitive numerical effort involved in the evaluation of the free surface integral appears to be a major obstacle in the analysis of second order problems in large water depth. It is, however possible to overcome this difficulty by making use of the expansion discussed above. An important aspect of the expansion g_m is that at $y_o=0$, the inconvenience caused by the finite integral vanishes. g_m then becomes a relatively simple expression and is well suited for the far field analysis. It also suggests that the first order potential ϕ can be similarly expanded on the free surface. For illustration, it can be shown that after making use of Green theorem one obtains the diffraction potential

$$\phi(\rho_o,\alpha_o,y_o) = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \epsilon_m \Psi^j_m(\rho_o,y_o) \left(A^j_m \cos m\alpha_o + B^j_m \sin m\alpha_o \right)$$

where the coefficients A_m^j and B_m^j can be obtained from the following body integrals in terms of the incident potential ϕ_i.

$$\begin{bmatrix} A^j_m \\ B^j_m \end{bmatrix} = \frac{1}{2\pi} \iint_{S_B} ds \left(\phi(\rho,\alpha,y) \frac{\partial}{\partial n} + \frac{\partial}{\partial n} \phi_i(\rho,\alpha,y) \right) \Lambda^j_m(\rho,y) \begin{bmatrix} \cos m\alpha \\ \sin m\alpha \end{bmatrix}$$

In addition, from table 1, it is significant that the infinite series in j converges very rapidly even when the distance between the observational point and the source point is relatively small. This property is of course desirable for reducing the upper bound of the 2D numerical integration area. In view of this and the possible series expansions of g_m and ϕ, a remedy for the efficient evaluation of the free surface integral is now possible. The methodology and formulation of the remaining part of this follows closely that of Ref 2., and will not be detailed here.

4 APPLICATION II - REMOVAL OF IRREGULAR FREQUENCIES

It is well known that the solution of the wave body interaction problems by means of integral equations may break down at a discrete set of irregular frequencies. These irregular values have no physical significance but correspond to the eigenvalues of the related interior Dirchlet problem. Various methods have been proposed to resolve the problem of irregular frequencies. Related work is the modified integral equation method proposed by Lee[3]. This method is believed to correspond to the distribution of additional singularities on the body surface. From this point of view, its numerical implementation in the general higher order boundary element codes presents difficulties, because it necessitates the use of more sophisticated methods to deal with the strongly singular integral equation. Ursell[4] demonstrated analytically how irregular frequencies may be removed by simply adding a suitable finite series of wave sources to the classical source potential. That treatment deals with two dimensional problems but the effectiveness of this method has not yet been demonstrated for three dimensional bodies of arbitrary geometries. Ursell's approach is reexamined in the present note. For simplicity, only the heaving motion of a unit hemisphere will be considered. Due to axisymmetry, the integral equation to be solved is

$$\pi\phi(p) + \int_{\partial L} \rho dl\ \phi(q) \frac{\partial}{\partial n} R_o(p,q) = -i\sigma \int_{\partial L} \rho dl\ n(q)\ R_o(p,q)$$

where R_o is the zero order ring source and n is the normal in the vertical direction. This integral equation is then solved by reducing it to a linear system of equations. Non-dimensional added mass (a_{33}) and damping (b_{33}) coefficients over a range of frequencies are depicted in figure 1. Here, the expected difficulty near the first irregular frequency (K≈2.56) can be clearly observed. This difficulty is removed by constructing a new source potential as follows

$$\tilde{R}_o(P,Q) = R_o(P,Q) + a_o\ \Psi_o^o(P)\ \Psi_o^o(Q)$$

where Ψ_o^o is defined as in section 2. For a suitable choice of a_o, it is evident from figure 1 that the difficulty associated with the first irregular frequency is removed, and away from it the agreement between the results obtained from the original source potential (a_o=0) and the modified source potential is very satisfactory. Thus, it has been shown that when the classical source potential is properly perturbed, a modified integral equation can be obtained for the same unknown function: this has been found not to break down over a range of frequencies associated with the heaving motion of a hemisphere. Nevertheless, it is clear that much work (both analytical and numerical) has to be done before this method could be implemented for the analysis of arbitrary three dimensional bodies.

5 REFERENCES

1 Martin,P.A. On the null-field equations for water-wave radiation problems, J.F.M., 1981, 113, 315-332
2 Chau,F.P. and Eatock Taylor,R. Second order velocity potential for arbitrary bodies in waves, 3rd Int. Workshop on Water Waves and Floating Bodies, Wood Hole, 1988
3 Lee,C.H. Removal of irregular frequencies using the modified integral equation, 3rd Int Workshop on Water Waves and Floating Bodies, Wood Hole, 1988
4 Ursell,F. Irregular frequencies and the motion of floating bodies, J.F.M., 1981, 105, 143-156

m=3, K=1, ρ=1, y=0, y_o=0			
ρ_o	EXPANSION	FINGREEN	N
1.05	1.668653	1.668664	87
1.10	1.214390	1.214393	49
1.20	0.787739	0.787740	29
1.50	0.347072	0.347072	15
2.00	0.158726	0.158726	10
2.50	0.099302	0.099302	8
3.00	0.068685	0.068685	7
3.50	0.045163	0.045163	6
4.00	0.022928	0.022928	6
5.00	-0.017810	-0.017811	5

Table 1

Comparisons of g_m at different radial distance. N is the total number of j terms used in the expansion.

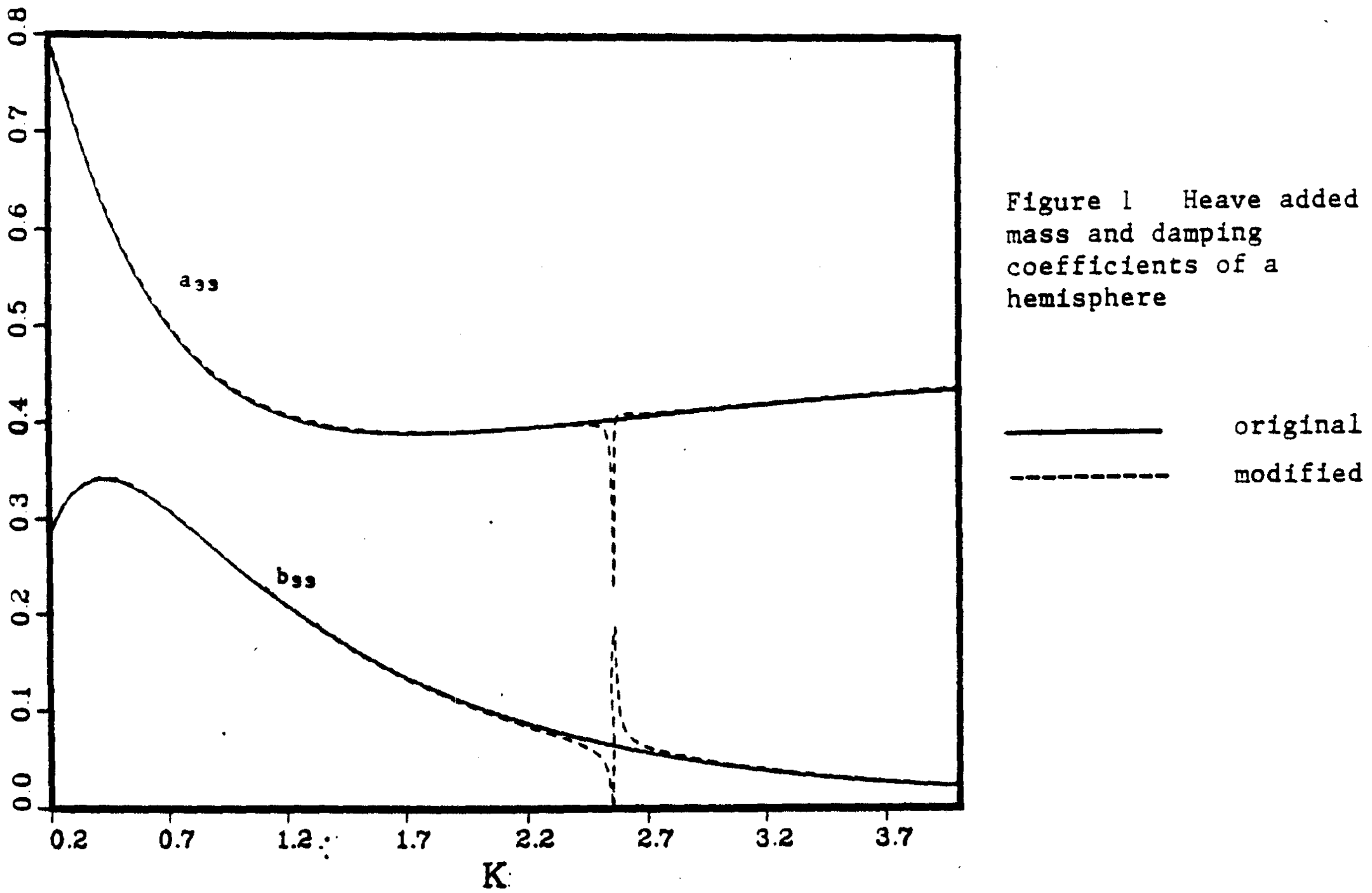

Figure 1 Heave added mass and damping coefficients of a hemisphere

DISCUSSION

Martin: A comment: P. Sayer (Proc.Roy.Soc. A, late 70's) used an integral-equation method for a heaving, half-immersed circular cylinder on water of constant finite depth, with the appropriate G. He proved that you can eliminate all irregular frequencies by adding a wave source located at the origin to G. This possibility was suggested by earlier work of Ogilvie & Shin.

Chau: In the present work, the same idea has been employed and demonstrated numerically that it works equally satisfactory for a heaving hemisphere.

Ursell: Your method is related to Havelock's, a different expansion for the wave source was used by Hulme who was able to calculate all the matrix elements in closed form. Recent work has shown that the approach of the truncated solution to the exact solution is slower than we would like; this is associated with the weak singularity at the free surface. It would be interesting to compare this with the integral-equation.

Chau: In the present expansion, the observational coordinates and the source coordinates are completely separated, and this may be considered as a generalization of the expansions given by Havelock and Hulme. Also, Hulme's expansion involves the differentiation of the associated Legendre function with respect to the degree, and this appears to be difficult to evaluate.