Is there an inconsistency in the treatment of low frequency second order vertical forces?

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Introduction

In the analysis of the low frequency behaviour of floating or tethered platforms, calculations are usually based on a second order Stokes expansion of the velocity potential, corresponding to a bichromatic undirectional wave with frequencies ω_1 and ω_2. It is found that the second order incident potential given by Bowers (1976) leads to a component of vertical force which, when one takes the limit $\omega_1 \to \omega_2$, appears to be different from the equivalent mean vertical force component in a regular wave. This phenomenon is investigated, leading to the suggestion that the second order set down in a bichromatic wave is often incorrectly stated.

Low frequency analysis

For water of depth d, Bowers' (1976) result is equivalent to

$$\Phi^{(2)}(t) = \mathrm{Re}\left\{ \sum_{i=1}^{2} \sum_{j=1}^{2} \phi_{ij}^{(2)} e^{i(\omega_i - \omega_j)t} \right\} \tag{1}$$

where

$$\phi_{ij}^{(2)} = -\frac{i g a_i a_j^*}{4} d_{ij} \frac{\cosh \bar{k}(z+d)}{\cosh \bar{k} d} e^{-i\bar{k}x}, \tag{2}$$

with

$$d_{ij} = \left[\frac{k_i^2}{\omega_i \cosh^2 k_i d} - \frac{k_j^2}{\omega_j \cosh^2 k_j d} + 2\bar{\omega}\frac{k_i k_j}{\omega_i \omega_j}(1 + \tanh k_i d \tanh k_j d)\right]\left[\frac{\bar{\omega}^2}{g} - \bar{k}\tanh \bar{k} d\right]^{-1} \tag{3}$$

and $\bar{\omega} = \omega_i - \omega_j$, $\bar{k} = k_i - k_j$. A term $\delta^{(2)}t$ is often added to the expression for $\Phi^{(2)}(t)$, with the constant $\delta^{(2)}$ determined from the assumption that the mean water level is at z=0.

The low frequency vertical force due to the second order incident wave is related to the second order elevation

$$\zeta^{(2)} = -\frac{1}{g}\left[\frac{\partial\phi^{(2)}}{\partial t} + \frac{1}{2}|\nabla\phi^{(1)}|^2 - \frac{1}{g}\frac{\partial\phi^{(1)}}{\partial t}\frac{\partial^2\phi^{(1)}}{\partial z\partial t}\right]\Bigg|_{z=0} \tag{4}$$

One finds that the component 12 of the low frequency set down wave is

$$a_{12}^{(2)} = \frac{1}{4}a_1a_2^*\left[-\bar{\omega}d_{12} + (k_1\tanh k_1d + k_2\tanh k_2d) - \frac{1}{g}k_1k_2(1+\tanh k_1d\tanh k_2d)\frac{g^2}{\omega_1\omega_2}\right]e^{-i(\bar{k}x+\bar{\omega}t)} . \tag{5}$$

We are concerned with the behaviour at very low difference frequencies. As $\bar{\omega} \to 0$ and $a_1 \to a_2 \to a/2$, the mean set down is given by

$$a^{(2)} = \frac{1}{4}\sum_{i=1}^{2}\sum_{j=1}^{2} -(\omega_i - \omega_j)d_{ij}\,a_ia_j^* - \frac{k}{2\sinh 2kd}\left(|a_1|^2 + |a_2|^2 + a_1a_2^* + a_2a_1^*\right)$$

$$= \frac{a^2}{4}\lim_{\bar{\omega}\to 0}(-\bar{\omega}d_{12}) - \frac{a^2k}{2\sinh 2kd} . \tag{6}$$

The second term on the right hand side of eq. (6) is exactly cancelled by the contribution from $\delta^{(2)}t$ which is often included in eq. (1). The purpose of this note, however, is to investigate the first term on the right hand side of eq. (6).

By taking the limit indicated, one finds that

$$\lim_{\bar{\omega}\to 0}(\bar{\omega}d_{12}) = \frac{kgC_g}{\omega^2}\,\frac{2\omega + kC_g\,\mathrm{sech}^2 kd}{C_g^2 - gd} = -K \tag{7}$$

where K is defined by eq. (7) and C_g is the group velocity, i.e.

$$C_g = \frac{1}{2}\frac{\omega}{k}\left(1 + \frac{2kd}{\sinh 2kd}\right) .$$

As a special case of this result, one can consider the case when each component wave is in moderately deep water, so that the approximations $kg = \omega^2$, $C_g = \omega/2k$ may be used. This then leads to

$$\lim_{\bar{\omega}\to 0} (\bar{\omega} d_{12}) = -\frac{1}{d} , \tag{8}$$

and the corresponding second order vertical force $F_3^{(2)}$ on a vessel having water plane area S_0 is then found to reduce to

$$F_3^{(2)} = \frac{\rho g a^2 S_0}{4d} . \tag{9}$$

The conclusion from equations (7) and (8) is that in the limit when the difference frequency tends to zero, the contribution to the mean vertical force from the second order potential is non-zero. But it is usually assumed that in a steady second order Stokes wave having a single fundamental frequency ω, there is no mean vertical force due to the second order potential (apart from the contribution associated with $\delta^{(2)}$ if this term is omitted from eq. (1)). The reason for this inconsistency is of course the limitation of the Stokes expansion under these conditions. As pointed out by Triantafyllou (1982), the potential $\Phi_{12}^{(2)}$ in eq. (1) can no longer be regarded as being of second order, when $\bar{\omega} \to 0$.

Regular wave analysis

Some insight into this problem can be gained from the regular wave analysis given by Mei (1983), based on a multiple scales expansion. He shows that in terms of the perturbation parameter $\epsilon = ka \ll 1$, the free surface elevation for a regular wave may be written

$$\zeta = \sum_{n=1} \epsilon^n \zeta^{(n)} \tag{10}$$

where

$$\zeta^{(1)} = \mathrm{Re}\left[a\, e^{-i(kx - \bar{\omega} t)}\right] \tag{11}$$

$$\zeta^{(2)} = b - \frac{ka^2}{2\sinh 2kd} + \frac{k\cosh kd\,(2\cosh^2 kd+1)}{8\sinh^3 kd} \mathrm{Re}\left[a e^{-2i(kx-\tilde{\omega}t)}\right] \tag{12}$$

$$\tilde{\omega} = \omega + \epsilon^2 a^2 \bar{\omega}_2 . \tag{13}$$

Here $\bar{\omega}_2$ is the second order frequency correction which is obtained from solution of the cubic Schrodinger equation for the slowly varying wave envelope. The parameter b is obtained by Mei as

$$b = b_0 - \frac{K}{4} a^2 \tag{14}$$

using our definition of the parameter K in eq. (7), and with b_0 equal to zero if the wave train is assumed to have started from rest at $x = -\infty$.

It is clear that the mean second order wave elevation given by eq. (12) is exactly equivalent to that obtained from the limit of eq. (6) as $\bar{\omega} \to 0$. The second order contribution to $\zeta^{(2)}$ from b is crucial. In the analysis presented by Mei for a regular wave, it is shown that the origin of this term is a slowly varying "first" order potential. But its form is only established by considering the Stokes expansion up to third order, and invoking a solvability condition. It appears that this term is often neglected.

Conclusion

To maintain consistency between the second order mean vertical force in a regular wave and the low frequency force in bichromatic waves, it is necessary to take account of a mean set down contribution $Ka^2/4$, where K is defined in eq. (7). If this term is neglected in the bichromatic wave analysis, a discontinuity appears on either side of the "diagonal" of the quadratic transfer function for second order vertical force. Analysis of a typical design has shown that this effect can be significant for a TLP.

References

Bowers, E.C. (1976) Trans. RINA 118, 181-191

Mei, C.C. (1983) The applied dynamics of surface waves, Wiley Interscience, pp 607-620.

Triantafyllou, M.S. (1982) J. Ship Res. 26, 97-105.

DISCUSSION

Palm: 1. In the Bernoulli eq. there is an arbitrary second order function of time, which is uniquely determined for a Stokes wave. If you have two waves tending towards each other, these functions may approach another limit than for the Stokes wave.

2. But the set down expressed by $\dot{\zeta}^{(2)}$ for a Stokes wave is not given by (4) if we mean the *physical* set down. Perhaps your limit for the set down is the correct physical one? That is easy to examine.

Eatock Taylor: The set down in a Stokes wave is $a^2k/(2\sinh 2kd)$, which is consistent with (4). This is a different term from the one causing the problem in bichromatic waves. The "arbitrary" term in the Bernoulli equation is related to the need to specify an initial condition, in order to arrive at the solution I have suggested (cf. Mei's book).

Grue: Say we measure the mean vertical force on a surface-penetrating cylinder in a wave-flume. First one turn on the wave generator with one frequency. Next one repeat the experiment with generating waves of two different frequencies, ω_1 and ω_2. Then we let $\omega_1 \to \omega_2$ in the experiment. 1. Will we measure the same uplift force in the two cases? 2, Which of the expressions for the force in the abstract is appropriate to compare with?

Eatock Taylor: It would be very worrying if the answer to Q1 were anything other than 'Yes'! The measured force should be compared with that given by the usual monochromatic wave theory, although of course various wave tank effects will cause important differences in the second order phenomena.

Yue: I am uncomfortable with your suggestion in the abstract that the regular wave second-order Stokes result, which is straightforward and unambiguous, should somehow be modifed based on a limit in two wave analysis, which must still strictly be valid only for small Ursell's parameter. Specifically, the permissive wave steepness must necessarily be bounded by the depth parameter, so that the contribution from the term in question must in fact be nil.

Eatock Taylor: The first point is agreed. As mentioned in my presentation the preliminary abstract was confusing on this point, but the final version attempts to clarify this. I do not believe that bounding the wave steepness has any influence on the point I am trying to make: the QFT is independent of wave amplitude.