Nonlinear surface waves at an underwater breakwater

John Grue

University of Oslo

We study the derformation of an incoming plane wave train propagating over an underwater breakwater, which is a horizontal box with axis parallel to the wave crests. The model is two-dimensional. The box is rectangular, standing on the sea bottom with its horizontal extension being long compared to the water depth in the shallow region above it. We study wave parameters where the flow in the shallow region is nonlinear. Elsewhere the flow is approximated by linear water wave theory. The mathematical formulation of the problem and the numerical implementation is transient.

This study is motivated by the beautiful phenomenon, which can be observed in a wave flume, that a wave train propagating over a slightly submerged body introduces higher harmonic free waves with appreciable amplitudes downstream of the body. In fact, the amplitude of the second harmonic free wave can be of same magnitude as that of the incoming wave.

Another principal motivation is the issue of matching between nonlinear and linear flow regimes.

Basic assumptions for the analysis below is that the fluid is inviscid, the motion irrotational and that there is no effects of viscosity.

1 Mathematical formulation

Let us introduce horizontal coordinate x and vertical coordinate y, with $y = 0$ at the still water level. We consider incoming deep water waves from $x = -\infty$ with wave length λ and amplitude A. Let h denote the water depth in the shallow region above the breakwater and let the vertical walls of the breakwater be located at $x = \pm L$. We shall consider situations where

$$h/\lambda \ll 1, A/h = O(1) \tag{1}$$

such that the flow in the shallow region is governed by the nonlinear shallow water equations

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(U(h+\eta)) = 0 \tag{2}$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + g\frac{\partial \eta}{\partial x} = 0 \tag{3}$$

Here η denotes the free surface elevation, U the vertically averaged horizontal velocity, t time and g acceleration of gravity.

We assume linearized potential flow outside the breakwater. A transient formulation is adopted due to presence of super harmonic wave components. Transmitted wave systems downstream as well as reflected wave systems upstream may then be described by vertical source distributions at $x = L, -h < y < 0$ and $x = -L, -h < y < 0$, respectively. The flow downstream is then governed by the velocity potential ϕ_L given by

$$\phi_L(x,y,t) = \frac{1}{\pi}\int_{-h}^{0} u_L(y',t)\ln\frac{r}{r_1}dy' - \frac{1}{\pi}\int_0^t d\tau \int_{-h}^{0} u_L(y',\tau)H_\tau(x,y,L,y',t-\tau)dy' \quad (4)$$

where $u_L(y',t)$ denotes the horizontal velocity of the flow at $x = L, -h < y < 0$, and H_τ is given by (see Yeung 1982)

$$H_\tau(x,y,x',y',t-\tau) = 2g\int_0^\infty \frac{dk}{\sqrt{gk}}\sin(\sqrt{gk}(t-\tau))\cos(k(x-x'))\exp(k(y+y')) \quad (5)$$

where

$$r = \sqrt{(x-x')^2+(y-y')^2} \quad (6)$$

and

$$r_1 = \sqrt{(x-x')^2+(y+y')^2} \quad (7)$$

The potential ϕ_{-L} for the reflected waves upstream is given with a corresponding source distribution at $x = -L$.

In the region $L - \epsilon < x < L$, with $\epsilon = O(h)$, we need to account for vertical accelerations of the fluid, thus (2) and (3) are unsuitable. We there adopt a linearized Green's theorem formulation of the flow with logaritmic Green function. Thus the velocity potential in this region is given by

$$\phi(x,y,t) = \frac{1}{\pi}\int_S(\phi\frac{\partial}{\partial n}\ln r - \ln r\frac{\partial\phi}{\partial n})dS \quad (8)$$

where S is the contour surrounding the volume $L - \epsilon < x < L, -h < y < 0$. At $y = 0$ $(L - \epsilon < x < L)$ ϕ and $\partial\phi/\partial n$ are linked by the linearized free surface condition. The corner flow at $x = -L$ is similarly handled.

2 Matching, start-up and far-field

The contour S is divided into segments, with ϕ and $\partial\phi/\partial n$ assumed constant on each panel. Matching between (4) and (8) then give u_L as unknown on $x = L, -h < y < 0$. At $x = L - \epsilon$ we assume that $\partial\phi/\partial n$ has no vertical variation. The time derivative of the combination of (4) and (8) then provide a relation between η and U at $x = L - \epsilon$, which is a radiation condition for the flow in the shallow region. Similar procedure applies for the construction of a radiation condition at the "upstream corner".

We assume that the motion upstream of the breakwater is composed of the transient reflected wave systems and a standing wave with potential ϕ_0 given by

$$\phi_0 = 2A\exp(ky)\cos(k(x+L))\sin(\omega t) \tag{9}$$

where

$$\omega^2 = gk = \frac{2\pi g}{\lambda} \tag{10}$$

For $t < 0$ we assume that the only motion of the fluid is the standing waves upstream. For $t > 0$ the water can flow over the breakwater governed by the equations described above. Products of the time simulations are the horizontal velocities $u_{\pm L}$ at $x = \pm L$, which entirely determine the transmitted waves, and together with (9) the reflected waves. As $t \to \infty$, $u_{\pm L}$ will oscillate with frequencies ω, 2ω, 3ω, ..., i.e.

$$u_{\pm L}(y,t) = \mathrm{Re}(u^{(1)}_{\pm L}(y)\exp(i\omega t) + u^{(2)}_{\pm L}(y)\exp(2i\omega t) + u^{(3)}_{\pm L}(y)\exp(3i\omega t) + \ldots) \tag{11}$$

The waves in the far-field can then be found from (4). For the surface elevation at $x = \infty$ we get

$$\eta(x,t) = Re(a^{(1)}\exp(i(kx-\omega t)) + a^{(2)}\exp(i(4kx-2\omega t)) + a^{(3)}\exp(i(9kx-3\omega t)) + \ldots) \tag{12}$$

where

$$a^{(n)} = \frac{n\omega}{g}\int_{-h}^{0} dy u^{(n)}_L(y)\exp(n^2k(y-iL)) \tag{13}$$

Main results of the simulations for nonbreaking waves are that $a^{(2)}$ predominate the super harmonic components downstream. $a^{(2)}$ may be comparable to $a^{(1)}$. Upstream the superharmonics are vanishingly small compared to the incoming waves.

References

[1] YEUNG, R. W. Transient heaving motion of floating cylinders. *J. Engg. Math. 16, pp. 97-119* 1982.

DISCUSSION

Ursell: I would have expected the incident wave to be almost completely reflected (because the incident wavelength is much greater than the depth above the breakwater). Why is this not the case?

Grue: For $\lambda \to \infty$ there is complete reflection. However, for e.g. $h/L=0.1$ and $\lambda/L<20$, the transmitted wave amplitude exceeds 0.78, surprisingly (h = local depth, L = horizontal length of breakwater, λ = wave length of incoming deepwater waves). In the example from the presentation, the parameters were: $h/L=0.1$, $\lambda/L=3$, $A/h<0.25$. For $A/h \sim 0.35$ the incoming waves break at the weather side of the breakwater, with complete reflection as consequence (experiments).

Akylas: Your numerical results indicate large slopes over the breakwater. Now, it is known that shallow-water theory is invalid when steep slopes are predicted. Perhaps a formulation in terms of a KdV equation, which includes some dispersive effects, would be more appropriate.

Grue: Dispersion effects may be important, and I will look into this. It is very simple to implement the full Bousinesq-equations in the model. The KdV-equation may not be appropriate since waves are travelling in both directions in the shallow region.

Tuck: Is there much effect of the vertical distribution of velocity in the apparent transmission wavemaker? If one was using a consistent matched asymptotic expansion approach for small depth/wavelength and depth/width ratios, the transmitted wave would appear as if it was generated by an isolated line source at the (shrinking) opening, i.e. the flux through the gap would be concentrated at a single point rather than distributed vertically.

Grue: The vertical variation of the flow at the corner is essential, and is thus important to account for. The works which Prof. Tuck has in mind account for this variation through a detailed corner-flow. The far-field waves do, however, look like as they were generated by sources at a single point.

Newman: What happens if you leave out the intermediate matching solution between the shallow and deep regions?

Grue: What I know is that the flow in the shallow region has very weak dependency on the vertical coordinate. This is, however, not true for the flow at the corner. Leaving out the intermediate matching solution between the shallow and deep regions may then result in loss of mass flux and/or energy. On the other hand, instead of matching surface elevation and mass flux, one can match mass flux and energy flux. This may very well work. I will investigate this.

Vinje: From my previous experience on this kind of problems I feel that you have to have some sort of buffer zone between the linear and the non-linear domains. In case you had done the matching at the actual interface I would expect that you would had run into more severe problems concerning reflection and/or numerical stability.

Grue: I will look into this in more detail.

Yeung: Concerning the point of short waves generated around the matching zone: I notice the heaving rectangular cylinder solution we obtained for the fully nonlinear problem (Yeung & Ananthakrishnan) in some occasions short waves propagating towards the center of the cylinder. It appears that the discontinuous depth can very well generate waves of a relatively short type, which may invalidate the use for shallow-water equations in the region above the cylinder.

Grue: I acknowledge this suggestion, and will look carefully to the origin of the short waves, which I get in the simulations. At this stage I suspect the short waves to be unphysical reflection from the matching of the flow at the lee-side corner of the breakwater. More study is needed here. I will look up results from John William's experiments and also look into results from new experiments by myself, and see whether these short waves are present or not.

Shallow-water theory may still be a good approximation to the flow in shallow region if the short waves have small amplitudes.

Wehausen: Are the experimental results in John William's dissertation consistent with yours?

Grue: So far, only preliminary experiments are done for visualizations. Results of the final experiments will be compared with John William's work (1965).