ITERATION FOR WATER WAVE INTEGRAL EQUATIONS

R.E. Kleinman
Center for the Mathematics of Waves
Department of Mathematical Sciences
University of Delaware
Newark, DE 19716, U.S.A.

At the Third International Workshop On Water Waves and Floating Bodies we showed how a stationary Richardson or one step over-relaxation method could be applied to the integral equations that arise in floating body problems with linearized free surface conditions. In particular we showed that if the spectrum of the integral operator is suitably confined then there always exists a relaxation parameter for which the iteration converges. Left open were a number of serious questions including:

1. Can we determine whether the spectrum is suitably confined without explicitly determining it for specific hull shapes so that existence of a relaxation parameter is assured?
2. Even if we know that such a relaxation parameter exists, can we find a suitable value for which the iteration converges?

The present paper presents positive answers to both of these questions. First it is shown that the spectrum of the integral operator for one of the integral equation formulations is confined in the desired fashion. Then we describe a successive over-relaxation method in which the relaxation parameter is updated at each iteration to minimize the residual error. In this algorithm the choice of the relaxation parameters is explicit and convergence of the sequence of iterates is proven.

Although the iteration method is equally applicable to a general class of integral operators we are only able to verify that the spectrum is suitably confined in the context of the particular boundary value problem under consideration. We will restrict our attention to the floating body problem in which we seek to determine a velocity potential, ϕ, in D_+, the domain bounded by a free surface ($y=0$), a flat bottom ($y=-h$), and the submerged portion of the hull(C_0). We seek a solution of Laplace's equation in D_+ which satisfies the radiation condition, the linearized free surface condition $\frac{\partial\phi}{\partial n}+k\phi=0$, has vanishing normal derivative on the bottom, and satisfies a Neumann condition $\frac{\partial\phi}{\partial n}=V$ on C_0, the ship hull. This problem is known to have a unique solution provided vertical rays from the free surface intersect the ship hull at most once.

The problem may be reduced to the well known integral equation

$$\phi(\mathbf{p})+\int_{C_0}\phi(\mathbf{q})\frac{\partial\gamma}{\partial n_{\mathbf{q}}}(\mathbf{p},\mathbf{q})ds_{\mathbf{q}}=\int_{C_0}V(\mathbf{q})\gamma(\mathbf{p},\mathbf{q})ds_{\mathbf{q}}\ ,\ \mathbf{p}\in C_0 \tag{1}$$

where γ is the Green's function which Fritz John derived for the domain without the ship. This equation we write in abstract form as

$$L\phi=v \tag{2}$$

where L is a bounded operator mapping $\mathcal{L}_2(C_0)\rightarrow\mathcal{L}_2(C_0)$. It is well known that this equation suffers from irregular frequencies which means that $L\phi=0$ has non-trivial solutions for some values of k hence (2) is not uniquely solvable for those values . Another way of saying this is that zero is in the spectrum of L, $0\in\sigma(L)$. One way to eliminate these irregular frequencies is to consider a different operator,

$$\phi(\mathbf{p})+\int_{C_0}\phi(\mathbf{q})\frac{\partial\gamma}{\partial n_{\mathbf{q}}}(\mathbf{p},\mathbf{q})ds_{\mathbf{q}}+\eta\int_{C_0}\phi(\mathbf{q})\frac{\partial\gamma}{\partial n_{\mathbf{q}}}(\mathbf{p},\mathbf{q})ds_{\mathbf{q}}=$$

$$\int_{C_0}V(\mathbf{q})\gamma(\mathbf{p},\mathbf{q})ds_{\mathbf{q}}+\eta\{\int_{C_0}V(\mathbf{q})\frac{\partial\gamma}{\partial n_{\mathbf{p}}}(\mathbf{p},\mathbf{q})ds_{\mathbf{q}}-V(\mathbf{p})\}\ ,\ \mathbf{p}\in C_0 \tag{3}$$

which we write in abstract form as

$$L_1\phi=v_1. \tag{4}$$

If $Im(\eta)>0$ then this equation is uniquely solvable, or $0\notin\sigma(L_1)$.

Now recall the result for solving this equation iteratively.

Theorem 1: If

$$\sup_{\lambda\in\sigma(L)}\{|\lambda|\}=M<\infty,\ \inf_{\lambda\in\sigma(L)}\{|\lambda|\}=m>0$$

and $0 \leq \theta_M - \theta_m < \pi$, where

$$\theta_M := \sup_{\lambda\in\sigma(L)} \{\arg(\lambda)\}, \; \theta_m := \inf_{\lambda\in\sigma(L)} \{\arg(\lambda)\}, ,$$

then there exists a complex number α such that $\sigma(I - \alpha L) < 1$ and the solution of (2) is given iteratively by

$$\phi_o \text{ arbitrary}, \; \phi_n = (I - \alpha L)\phi_{n-1} + \alpha v, \; n \geq 1 . \tag{5}$$

If k is an irregular frequency then $m = 0$ and we must replace L by L_1.

We now outline a proof of the following theorem which shows that the floating body integral equations can be solved in this way.

Theorem 2: $\sigma(L_1)$ satisfies the conditions of Theorem 1 and if k is not an irregular frequency then $\sigma(L)$ also satisfies the conditions of Theorem 1.

Proof: We will confine our remarks to the latter case and consider only eigenvalues. First note that since L is bounded, $|\lambda| \leq \|L\|$ for all λ such that $L\phi = \lambda\phi$ has nontrivial solutions (eigenfunctions). Moreover $\lambda = 0$ is not in $\sigma(L)$. This enables us to show that appropriate M and m exist. To show that the spectrum is confined to a wedge shaped domain we proceed as follows. Assume that $L\phi = \lambda\phi$. Then define

$$\int_{C_0} \phi(\mathbf{q}) \frac{\partial \gamma}{\partial n_{\mathbf{q}}}(\mathbf{p}, \mathbf{q}) ds_{\mathbf{q}} = u_{e,i}, \quad \mathbf{p} \in D_{+,-} \tag{6}$$

where D_- is the interior of the ship below the water line (bounded by C_0 and the projection of C_0 in the free surface). Then using the well known jump conditions for double layer potentials we find that

$$u_e = L\phi - 2\phi = (\lambda - 2)\phi \;\text{ on } C_0 \tag{7a}$$

$$u_i = L\phi = \lambda\phi \;\text{ on } C_0. \tag{7b}$$

These equations together with continuity of the normal derivative of the double layer enable us to deduce that on C_0

$$\lambda u_e = (\lambda - 2) u_i \tag{8a}$$

$$\frac{\partial u_e}{\partial n} = \frac{\partial u_i}{\partial n} . \tag{8b}$$

Then u_e and u_i satisfy a homogeneous transmission problem for Laplace's equation. Note that $\frac{\partial u_e}{\partial n} = 0$ on $y = -h$, u_e and u_i satisfy the linearized free surface condition and u_e satisfies the radiation condition since these properties are inherited from the Green's function. Now a straightforward but lengthy reproduction of the standard uniqueness proof yields the fact that u_e and u_i , hence ϕ, must vanish if $Im(\lambda) \geq 0$. Hence a necessary condition for $\lambda \in \sigma(L)$ is $Im(\lambda) < 0$. This together with the previous bounds on $\|\lambda\|$ suffice to establish the desired result.

This means that the conditions for the applicability of Theorem 1 are fulfilled and there exists an α such that the iteration scheme outlined in (5) may be applied. Finding such an α is the next task. Previously we proposed that α be chosen to minimize the residual error in the first iterate with $\phi_0 = 0$. That is , choose α to minimize $\|v - \alpha L v\|$ which leads to an explicit value, $\alpha = \frac{(v,Lv)}{\|Lv\|^2}$. This choice has been shown to yield useful numerical results but numerical experiments have shown that it is not optimal and indeed does not suffice to give to give convergent results in some extreme cases where convergence could be obtained with a better choice.

There is another iterative method, a successive over-relaxation method wherein a new relaxation parameter is defined at each iteration step to minimize the residual error. Specifically it consists of the algorithm

$$\phi_o \text{ arbitrary }, \ \phi_n = (I - \alpha_n L)\phi_{n-1} + \alpha_n v, \ n \geq 1 . \tag{9}$$

where

$$\alpha_n = \frac{(\phi_{n-1}, L\phi_{n-1})}{\|L\phi_{n-1}\|^2} . \tag{10}$$

This algorithm is shown to converge for all cases when not only does there exist an α such that $\sigma(I - \alpha L) < 1$, i.e. Theorem 1 holds, but in addition that $\|I - \alpha L\| < 1$. If this stronger condition holds then the iteration scheme in (9,10) converges to the desired solution. The appealing feature is that the choice of relaxation parameter at each step is completely determined.

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DISCUSSION

Miloh: When dealing with a Fredholm integral equation of the second kind there is a very efficient method for solving iteratively the I.E and that is by substracting first the transpose kernel and using the Gauss flow theorem with $\alpha=1$ in the Neumann scheme. However this procedure fails when solving a Fredholm integral equation of the first kind, probably due to Picard's theorem which implies that for thin type of I.E a general solution exists only in the Lebesgue sense and the numerical iteration scheme may converge unto a solution which may be ruled out based on physical grounds. Can you please comment on this problem?

Kleinman: To solve $Lu=v$ when L is a first kind equation (or even a second kind equation with unempty null space i.e. non trivial solutions of the homogeneous equation) I believe that the iteration $u_n=u_{n-1}+\alpha_n v_n$ will still converge and converges to a function which numerizes the quantity $\|v-Lu\|$ for a varying over the space H. I think it is also true that the convergence will be much slower than when $\|L^{-1}\|$ is bounded. (In the usual case of 1st kind equations the operator L is often compact in which case L^{-1} is unbounded.)