ABSTRACT

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A NOTE ON THE SHALLOW WATER WAVE THEORY

At the workshop in Bristol two years ago I presented an equation for linearized surface waves which was given the name "The Ekofisk Equation":

$$\sum_{n=0}^{\infty} \frac{(-1)^n h^{2n}}{(2n)!} \left\{\frac{h}{2n+1} \nabla^{2n+2}\eta + \left(1 - \frac{\omega^2 h/g}{2n+1}\right) \nabla h \cdot \nabla(\nabla^{2n}\eta) + \frac{\omega^2}{g} \nabla^{2n}\eta\right\} = 0$$

h - water depth

ω^2/g - deep water wave number

η - surface elevation (without the time factor $e^{-i\omega t}$)

∇ - the gradient operator in the two horizontal coordinates x and y.

The equation is a partial differential equation for the surface elevation. The three dimensional water wave problem has been reduced to two dimensions. The penalty is the infinite order of the equation.

However, when the depth is small, the above series can be truncated. We thereby obtain shallow water wave theories, the accuracy of which depends upon the number of retained terms in the expansion.

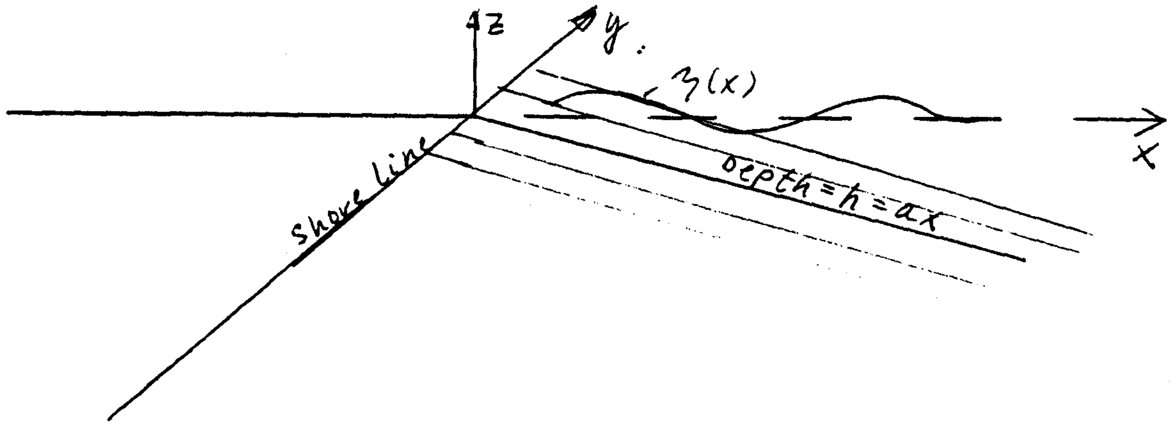
If we retain only one term we obtain the following equation:

$$h\nabla^2\eta + \left(1 - \frac{\omega^2}{g} h\right) \nabla h \cdot \nabla\eta + \frac{\omega^2}{g} \eta = 0 \qquad \text{eq. 1}$$

For comparison we also write down the equation that has become standard in shallow water wave theory ("The Long Wave Equation"):

$$h\nabla^2\eta + \nabla h \cdot \nabla\eta + \frac{\omega^2}{g} \eta = 0 \qquad \text{eq. 2}$$

The purpose of this contribution to the workshop is to show that the two equations can give markedly different results unless $|\nabla h|$ is also small, and that eq. 1 is the correct equation to use. We compare the two by means of a classical special case: Edge waves (Ursell)



The two equations now read:

$$a x \nabla^2 \eta + \left(1 - \frac{\omega^2}{g} a x\right) a \frac{\partial \eta}{\partial x} + \frac{\omega^2}{g} \eta = 0 \qquad \text{eq. 1}$$

$$a x \nabla^2 \eta + a \frac{\partial \eta}{\partial x} + \frac{\omega^2}{g} \eta = 0 \qquad \text{eq. 2}$$

We seek solutions as edge wave modes of the form

$\eta = e^{i \beta y} f(x)$

β - wave number in the y-direction

i.e.:

$$a x \, f''(x) + \left(1 - \frac{\omega^2 a}{g} x\right) a \, f'(x) + \left(\frac{\omega^2}{g} - a\beta^2 x\right) f(x) = 0 \qquad \text{eq. 1}$$

$$a x \, f''(x) + a \, f'(x) + \left(\frac{\omega^2}{g} - a\beta^2 x\right) f(x) = 0 \qquad \text{eq. 2}$$

Both equations lead to solutions in terms of confluent hypergeometric functions.

f(x) must also satisfy the following requirements:

i) $f(0) = \text{finite}$

ii) $f(x) \to 0$ as $x \to \infty$

This has two consequences:

The confluent hypergeometric functions are specialized to Laguerre polynomials (L_n).

Edge wave modes occur when certain relations are satisfied between β and ω.

The solutions are:

Eq. 1:

$$f(x) = e^{-\alpha x} L_n\left(\frac{\omega^2(2+a^2)}{ga(2n+1)} x\right)$$

$$\alpha = \frac{\omega^2(1-na^2)}{ga(2n+1)} \quad ; \quad \beta = \frac{\omega^2}{ga(2n+1)} \sqrt{1+a^2-(n^2+n)a^4}$$

n = 0, 1, 2.... <u>restricted by $n < 1/a^2$</u>

Eq. 2:

$$f(x) = e^{-\alpha x} L_n\left(\frac{2\omega^2}{ga(2n+1)} x\right)$$

$$\alpha = \frac{\omega^2}{ga(2n+1)} \quad ; \quad \beta = \frac{\omega^2}{ga(2n+1)}$$

n = 0, 1, 2.... <u>unrestricted</u>

We observe that there is a marked difference in behaviour of the two solutions. The results are not only numerically different. They are also qualitatively different:

Eq. 1 predicts a finite number of edge wave modes.

Eq. 2 predicts an infinite number of such modes.

In this respect eq. 1 is in agreement with the exact theory, whereas eq. 2 is not.

By direct comparison between the two solutions we see, however, that they approach each other when $a = |\nabla h| \to 0$.

The edge wave example illustrates the following:

CONCLUSION

To within the first order in water depth, h, eq. 1 is the correct shallow water wave equation. This conclusion follows directly from the "Ekofisk equation". Eq. 2 is only consistent to this order if $|\nabla h|$ is small in the same sense as h, i.e.: $|\frac{\nabla h}{h}| = O(1)$.
The situation is somewhat paradoxical since the equation (eq. 2) that has become standard in shallow water wave theory can break down when $h \to 0$, as we just saw in the edge wave example where $h(0) = 0$.

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Recorded discussion after Mehlum's presentation

Discusser: E. O. Tuck

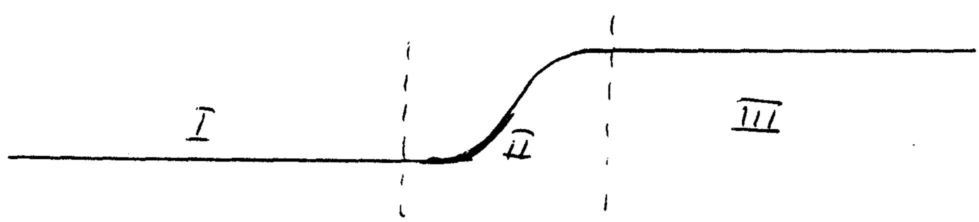
If L = horizontal length scale = wavelength scale, then the bracket factor

$$\left(1-\frac{\omega^2 h}{g}\right) = 1 + 0(h^2/L^2)\left(\text{since } \omega^2/g \rightarrow k^2 h = \frac{4\pi^2 h}{\lambda^2} \text{ in shallow water theory}\right).$$

Hence is the extra term not small, $(=b^2/L^2)$ times terms that are retained?

Authors reply:

Consider a bottom profile like the following:



In sections I and III the two "Long Wave" equations (1 and 2) of the abstract give the same results (to first order). In section II there is not a well defined wavelength that can be used in order arguments. Indeed, one may ask whether there is a wave there at all. The claim is that eq. 1 is uniformly valid throughout without the need to "match in" a special solution in section II.

In actual examples like the one above it is found that the term:

$\frac{\omega^2}{g} h \nabla h \nabla \eta$ is as important as $h \nabla^2 \eta$ which is retained in both equations.

Discusser: T.R. Akylas

i) In deriving the shallow–water equation, the author assumes that $h \ll 1$; so, it seems inconsistent to keep the 0(h) terms in the expression $1 - \frac{\omega^2}{g} h$

ii) Aslo, for a beach of uniform slope, the shallow-water assumption fails far from the shoreline; this is true for both shallow-water equations derived by the author. So, it seems coincidental that a finite number of edge waves are predicted.

Authors reply:
i) See reply to E.O. Tuck.

ii) The edge wave illustration used in the abstract is not appropriate. The point here was to show the difference from a well published example: The approximate treatment of edge-waves given in C.C. Mei: The applied Dynamics of Ocean Surface Waves, Section 4.8. The excuse for using shallow water theory for this purpose is that edge waves are local phenomena travelling along the shore line. But the illustration is still not appropriate. The predicted cutoff is due to the change of sign of the factor $1 - \omega^2 h/g$ which is certainly not consistent with the shallow water assumption. This was pointed out by Thor Vinje.

Discusser: G. Pedersen
You have compared your eq. 1 to the hydrostatic shallow water equation. However, if you carry out the long wave expansion one step further (to give a Boussinesq equation) you will find several correction terms. I belive that one of these terms corresponds to extra terms of eq. 1 respective to the shallow water equation. The question is whether or not this particular term is the most significant correction term, which I in general doubt. You should also perform a quantitative comparison with full potential theory - even though you find a bound on the number of eigenmodes that is different from what is known from the litterature.

Authors reply:
The terms you refer to are precisely the terms obtained if one more term is retained from the full expansion given in the start of the abstract (The Ekofisk equation.). I belive that the extra term in eq. 1 is the most significant correction term. This belief is based partly on the way the Ekofisk equation is deduced (a Taylor expansion in h) partly on numerical comparisons. But I have not proved that later terms from the full equation in sum will not give as significant correction terms as the one I have retained in eq. 1. This is still an open question.

Concerning your comment on the eigenmodes, I agree. See also the reply to T.R. Akylas.