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AN ASYMPTOTIC SOLUTION OF THE SECOND-ORDER DIFFRACTION PROBLEM

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Complete second-order computations for wave diffraction by axisymmetric bodies reveal that the second-harmonic component of the pressure field is particularly significant at large depths. For large offshore structures such as tension-leg platforms, which are restrained by taut moorings, the resulting contribution to the vertical pressure force and mooring loads may dominate the first-order linear component.

The existence of a second-harmonic pressure at large depths is well known in the theory of two-dimensional standing waves, where this second-order component is independent of depth. The same phenomenon can be expected for a partial standing wave, resulting from wave reflection by a two-dimensional body. Since the transmitted wave lacks such a component there is a transition in the horizontal direction at large depths from the $O(1)$ pressure far upstream to the exponentially small pressure downstream. This suggests that the second-harmonic pressure in the vicinity of the diffracting body may be more important at large depths than the first-order exponential component.

A similar result can be expected in three dimensions, where the inhomogeneous free-surface boundary condition forces a non-radiating second-order solution along the ray directly opposite to the propagation angle of the incident wave. This singular feature of the far-field asymptotic solution was noted by Molin, and has been discussed extensively in connection with the appropriate statement of the second-order radiation condition.

This discussion suggests that a connection exists between the far-field behavior of the quadratic forcing function in the second-order free-surface condition and the slow rate of attenuation with depth of the resulting near-field solution. It also suggests that an analysis which considers only the far-field forcing function may lead to a relatively simple second-order solution which approximates the dominant second-order pressure field acting on the body.

TWO-DIMENSIONAL ANALYSIS

In two dimensions the second-order free-surface condition takes the form

$$\phi_{2z} - 4K\phi_2 = q(x), \tag{1}$$

where K is the first-order wavenumber and the 'quadratic transfer function' $q(x)$ involves products of the first-order solution ϕ_1. A particular solution ϕ_2 can be derived formally by interpreting $q(x)$ as an oscillatory pressure imposed on the free-surface, or simply from Fourier transformation, with the result

$$\phi_2 = \frac{1}{\pi}\int_{-\infty}^{\infty} q(\xi)d\xi \int_0^{\infty} \frac{\cos k(x-\xi)}{k-4K}\, e^{kz}\, dk. \tag{2}$$

Here the contour of integration is deformed in an appropriate manner to conform with the radiation condition.

If the depth $|z|$ is asymptotically large, and exponentially small terms are neglected, the leading-order contribution to (2) must come from the vicinity of $k = 0$, and if the integral of $q(x)$ over the complete free surface exists and is finite, (2) is clearly $O(z^{-1})$. However, when products of the first-order diffraction potential are considered it follows that while $q(x)$ vanishes far downstream, at a sufficient rate for the above estimate to apply, $q(x)$ approaches a constant value $-4KC$ far upstream in the direction opposite to the incident wave. This constant is proportional to the product of the incident and reflected wave amplitudes; with this normalization C is the non-wavelike constant in the potential ϕ_2 far upstream, in accordance with (1).

Since the local features of $q(x)$ give a contribution inversely proportional to the depth, the leading contribution to the potential can be derived from the simplified quadratic forcing function

$$q(x) = -4KC\,H(x)\,. \tag{3}$$

Here $H(x)$ denotes the Heaviside step function and the horizontal coordinate is defined such that the waves are incident from $x = +\infty$. Using generalized function theory to interpret the contribution from (3) to (2), and assuming that the field point $x = O(1)$, the integral over the free surface is equivalent to $-4\pi KC\delta(k)$ where δ denotes the Dirac delta-function. On this basis it follows that

$$\phi_2 = \frac{1}{2}C + O(z^{-1})\,. \tag{4}$$

Thus in the two-dimensional case the dominant pressure at large depths near the body is a constant, equal to precisely half of the corresponding constant pressure in the upstream partial standing wave, or the mean of the pressures far upstream and downstream. For sufficiently large depths this second-order pressure clearly will dominate the corresponding first-order component.

THREE-DIMENSIONAL ANALYSIS

In terms of cylindrical polar coordinates (r, θ, z), with $\theta = 0$ the incident-wave direction, the far-field approximation of the quadratic forcing function is defined as

$$q(r,\theta) \sim F(\theta)(Kr)^{-1/2}\, e^{-iKr(1+\cos\theta)}\,, \tag{5}$$

where $F(\theta) = -2iK^2A^2 f(\theta)(1-\cos\theta)$ and $f(\theta)$ denotes the angular dependence of the first-order scattered-wave amplitude in the far field. Note that $F(0) = 0$, in accordance with the fact that the scattered wave is purely progressive along the ray $\theta = 0$, whereas the forcing function is non-zero and non-oscillatory along the ray $\theta = \pi$.

A particular solution analogous to (2) can be derived using Fourier-Bessel transforms, in the form

$$\phi_2 = \frac{1}{2\pi}\iint q\,dS \int_0^\infty \frac{k}{k-4K}\, e^{kz} J_0\!\left(k\sqrt{r^2+\rho^2-2r\rho\cos(\theta-\alpha)}\right) dk\,. \tag{6}$$

The surface integral is over the free surface, i.e. the portion of the plane $z = 0$ exterior to the body.

An asymptotic expansion of (6) can be derived for large $|z|$, by expanding the factor $k/(k-4K)$ in powers of k and integrating term-by-term. Neglecting contributions which are exponentially small, this leads to a representation of (6) in terms of vertical dipoles and higher-order multipoles on the free surface. The leading-order contribution, of order z^{-1}, is

$$\phi_2 \sim -\frac{z}{8\pi K}\iint [R^2+\rho^2-2r\rho\cos(\theta-\alpha)]^{-3/2} q\, dS. \tag{7}$$

where $R^2 = r^2 + z^2$ denotes the spherical radius from the origin. Proceeding as in the two-dimensional analysis, the far-field approximation (5) may be substituted in (7), and integrated over the entire plane $z = 0$,

$$\phi_2 \sim \frac{z}{8\pi K^{3/2}}\int_0^\infty \rho^{1/2}d\rho \int_0^{2\pi} F(\alpha)\exp(-iK\rho(1+\cos\alpha)) [R^2+\rho^2-2r\rho\cos(\theta-\alpha)]^{-3/2}d\alpha. \tag{8}$$

If the variable ρ in (8) is re-scaled in terms of R, the argument of the exponential function is proportional to the large parameter KR. Thus the second integral in (8) can be evaluated by the method of stationary phase, with the result

$$\phi_2 \sim \frac{F(\pi)\,e^{i\pi/4}z}{4(2\pi)^{1/2}K^2}\int_0^\infty d\rho[R^2+\rho^2+2r\rho\cos\theta]^{-3/2}. \tag{9}$$

There is no contribution from the stationary point at $\alpha = 0$, since $F(0) = 0$. After evaluating the last integral it follows that

$$\phi_2 \sim \frac{F(\pi)\,e^{i\pi/4}}{4(2\pi)^{1/2}K^2}\,\frac{z}{R(R+z)}. \tag{10}$$

For field points close to the vertical axis,

$$\phi_2 \sim \frac{F(\pi)\,e^{i\pi/4}}{4(2\pi)^{1/2}K^2}\,\frac{1}{z}. \tag{11}$$

To test the approximations derived above, comparisons are shown in Figures 1 and 2 with the complete numerical computations by Kim and Yue (1988) of the second-order pressure distribution and vertical force acting on a circular cylinder of radius a and draft $d = 4a$. These comparisons indicate that our approximation is useful in describing the dominant part of the pressure distribution and vertical force acting on an axisymmetric cylinder, at depths comparable to or larger than the radius. It remains to be shown that useful predictions can also be made for non-compact bodies, such as tension-leg platforms, where exact numerical predictions are not available for comparison.

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Molin, B. 1986 Second-order double frequency loads and motions for 3D bodies. *Proc. First International Workshop on Water Waves and Floating Bodies*, MIT.

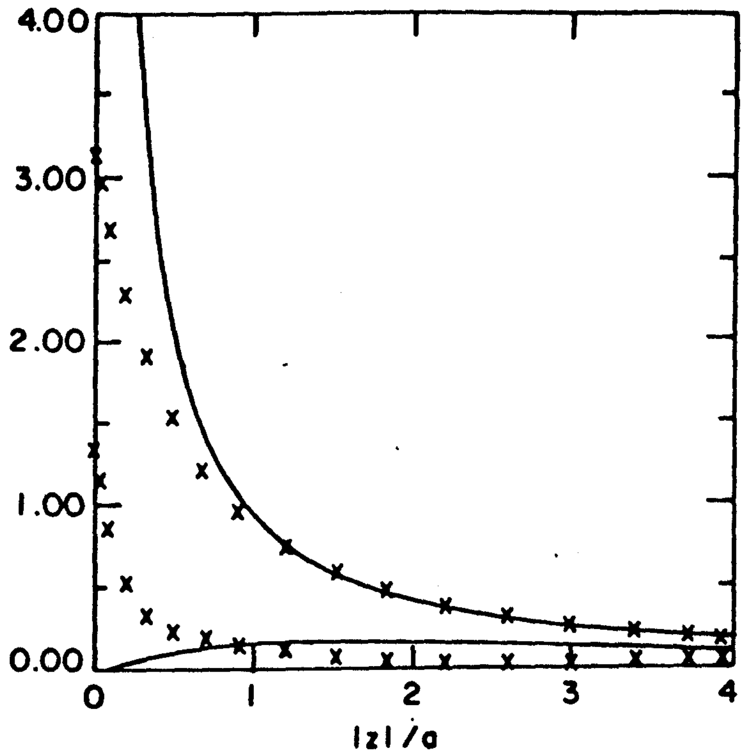

Figure 1. Approximation for the second-order, second-harmonic pressure distribution on the vertical axisymmetric cylinder as a function of depth, on the weather size ($\theta = \pi$, upper curve) and the lee size $\theta = 0$, lower curve). These results are for a cylinder of draft $4a$, where a is the radius, and $Ka = 1.52$. The pressure is normalised by the quantity $\rho g A^3/a$. The asterisks denote the corresponding results from the numerical solution by Kim and Yue (1988).

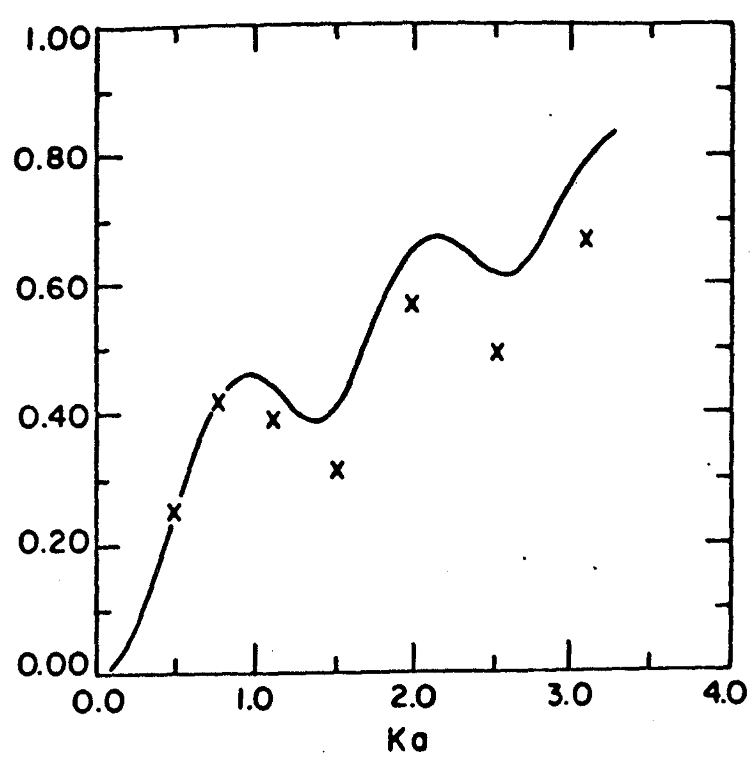

Figure 2. Second-order, second-harmonic vertical force on the cylinder (see Figure 1). The force is normalised by the quantity $\rho g A^2$. The asterisks denote the corresponding results from the numerical solution by Kim and Yue (private communication).

DISCUSSION

Natvig: Your work is for unidirectional waves. It would, however, be of great interest to know what the effect on the sum-frequency forcing directional waves would have? Is it likely that short crested seas could worsen the problem of springing exitations?

Newman: I have not made calculations for short-crested seas but this appears to be a straightforward extension of the asymptotic analysis, with the total force proportional to a linear combination of the first-order scattering amplitude function of each separate incident wave component. It would be very interesting to see the results of this extension.