Spectral Boundary Integral Methods for Gravity-Capillary Waves

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Longuet-Higgins and Cokelet (1976) analyzed breaking of two-dimensional steep gravity waves with a periodic boundary condition using a boundary integral method derived from Green theorem. Vinje and Brevig (1981) formulated a boundary integral method from Cauchy integral theorem. Baker, Meiron and Orszag (1982) and Roberts (1983) used spectral vortex formulations. These results often have unphysical numerical instabilities, especially when surface tension is included, making it difficult to analyze the stability of nonlinear waves.

Our two-dimensional gravity-capillary wave simulations using a spectral boundary integral method, are very accurate when the contour geometry is smooth and nodal spacing is uniform. We assume that the fluid is inviscid, incompressible and the flow is irrotational. Since the complex potential β is analytic inside the fluid domain, Cauchy's theorem gives

$$\oint \frac{\beta(s)}{z(s)-\zeta_k}\frac{dz}{ds}ds = i\pi\beta(\zeta_k) , \tag{1}$$

where s is an arclength parameter that follows the material points, and dz/ds is evaluated spectrally using a fast Fourier transform routine. The algebraic system is formed by discretization of the integral and letting the kernel singularity approach each of the N nodal points, $\zeta_k \rightarrow z_k$. The integral equation gives $2N$ real algebraic equations for N real unknowns. Schultz and Hong (1988) showed that the overdetermined algebraic system is more accurate than the strong and weak algebraic systems in most cases. Huh and Schultz (1989) showed that the spectral boundary integral methods give better accuracy than piecewise-linear approximation method. Here, two spectral formulations are outlined. One is based on the integration at every other node point to avoid the kernel singularities by Baker (1983) (spectral I), and the other is based on the kernel desingularization by Roberts (1983) (spectral II). After descretization, the algebraic equations are

$$\sum_{j=1}^{N} \Gamma_{jk} \beta(s_j) \; = \; 0 \qquad \text{for } k = 1 \ldots, N \; . \tag{2}$$

In spectral I, the influence coefficients Γ_{jk} are

$$\Gamma_{jk} = \begin{cases} \frac{1}{z(s_j)-\zeta_k}\left(\frac{dz}{ds}\right)_j & \text{if } |j-k| \text{ is odd} \\ -i\pi & \text{if } j = k \\ 0 & \text{if } |j-k| \text{ is even and } j \neq k \; . \end{cases} \tag{3}$$

In spectral II, Γ_{jk} are now

$$\Gamma_{jk} = \begin{cases} \frac{1}{z_j - z_k}\left(\frac{dz}{ds}\right)_j + \frac{dC_j}{ds}(s_k) & \text{if } j \neq k \\ -\sum_{i=1, i\neq k}^{N} \frac{1}{z_i - z_k}\left(\frac{dz}{ds}\right)_i & \text{if } j = k \; . \end{cases} \tag{4}$$

Here the cardinal function C_j is

$$C_j = \frac{1}{N} \sin \pi (s - s_j) \cot \frac{\pi}{N}(s - s_j) \; . \tag{5}$$

If ϕ and the spatial coordinates x and y are given on the free surface, ψ can be calculated from these algebraic equations. Then from the fully nonlinear kinematic and dynamic free surface condition, the time derivative of x, y and ϕ are obtained. From these time derivatives and given values of x, y and ϕ, new values of x, y and ϕ can be estimated. We solve the algebraic equations with a conjugate gradient iterative technique, and time march with a fourth order predictor-corrector scheme.

We present spectral boundary integral methods for several test cases: gravity-capillary waves of the permanent form as computed by Schwartz and Vanden-Broeck (1979). The figure shows the typical results of a piecewise-linear and the spectral II method. Adding surface tension with these methods shows promise for further study of Rayleigh-Taylor instability and Kelvin-Helmholtz instability which will now be well posed. The improved algorithm will also be used for studying short and long wave interaction, where a nodal point redistribution scheme should be developed since the nodal spacing becomes very uneven after long time.

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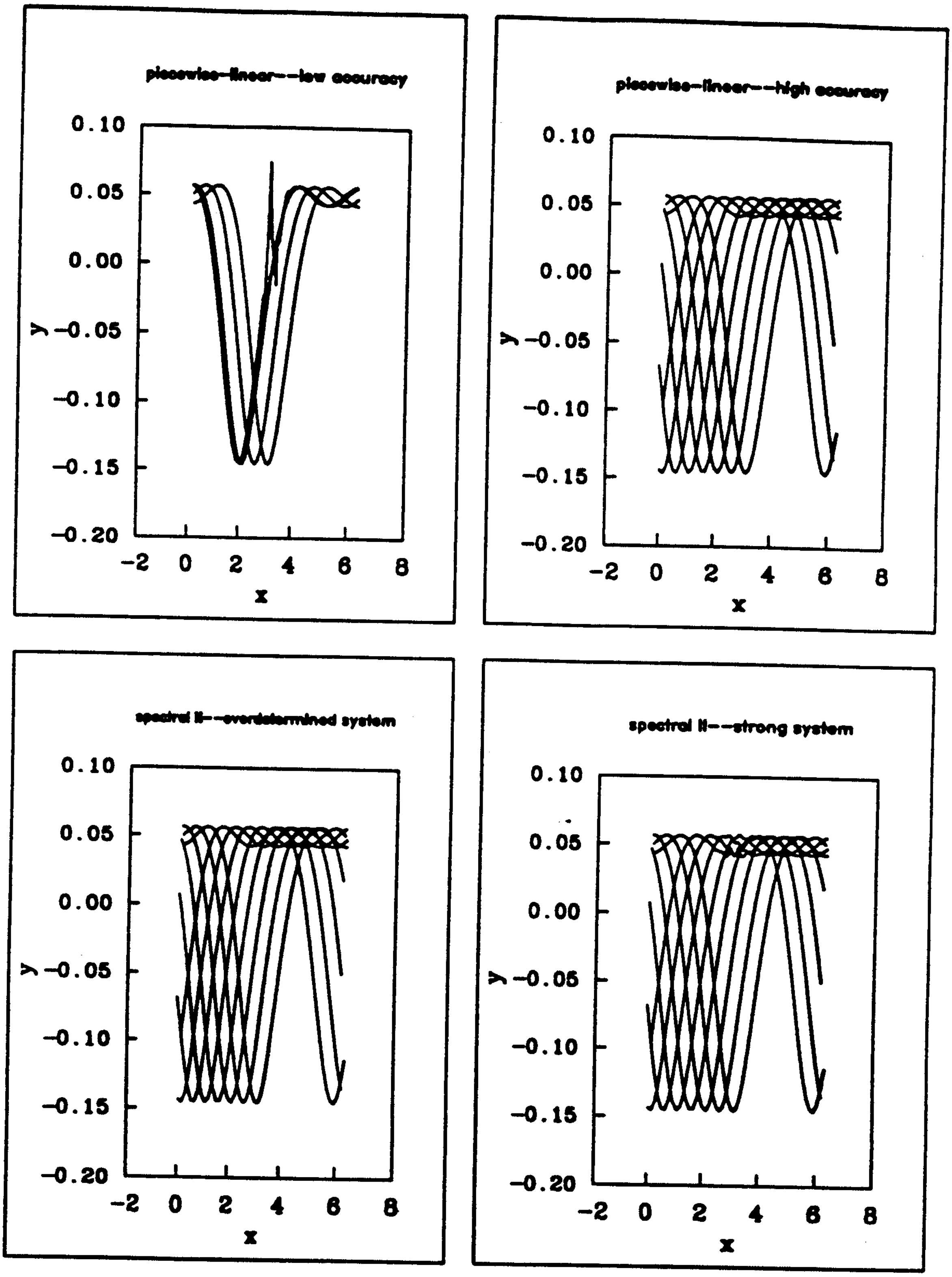

Piecewise-linear and spectral approximations of gravity-capillary waves

DISCUSSION

Van Hoff: I understand that the spectral method is based on $a=F(s)$, where a = complex amplitude, s is spatial coordinate. FFT is used if $s_i=s_j$, $i \neq j$. During evolution of solution, s_i tends to $s_i \neq s_j$. Isn't it possible to define say $\{t_i\}$, $t_i=t_j$, $i \neq j$. Such that $s_i=\text{Re}F(t_i)$. Then a can be expressed as $a=U(s;t)F(t)^*$ where $U(s;T)$ is a form of transfer function, possibly in terms of Δx, Δy in the form $\Delta s=\Delta x+i\Delta y=\Delta r\ e^{i\Delta\varphi}$. This enables a to be directly expressed as $a=U^{-1}F(t)$. Updates in regridding is then via *. This might work?!

Schultz & Huh: We are not sure if we understand the details of your suggestion, however s is a material parameter, not a physical coordinate in our formulation. Hence s is evenly spaced at every time step. We feel that regridding may be necessary at each time step. So using a FFT mapping in this process may be more efficient than our present method.

Grilli: After working on similar problems, but without surface tension, we have reasons to believe that the time marching procedure influences the spatial stability of the solution. In particular, formulations based on predictor-corrector methods, that are essentially one-point methods in space seem to lead to saw-tooth instability, whereas those based on Taylor expansions, that account for the variation of the solution along the boundary, seem to be more stable. What is your opinion about that in the context of the spectral method you have been using?

Schultz & Huh: At first thanks for your suggestion. You suggest a Taylor series expansion (Dold and Peregrine, 1984) instead of Hamming's predictor-corrector or Runge-Kutta schemes. Since we are trying to use Taylor expansion right now, we cannot now say whether it is more stable than our time marching procedure. We hope we can show the Taylor expansion results in the near future. At any rate we would not expect it to be any more accurate.