Second order deformation of the free–surface around a vertical cylinder. Part II

Y-M Scolan B. Molin
Institut Français du Pétrole

At the third workshop B. Molin and L. Boudet [1] proposed a method to calculate the second-order diffraction potential around a vertical cylinder; the purpose of the study being the extension to the case of multiple cylinders. Even though this method provided good results concerning the second–order pressures and loads on the cylinder, some difficulties appeared for the correct modeling of the second-order free–surface elevation. The reasons are essentially related to the discretization of the free–surface which must be refined enough but also determines the size of linear systems to solve. We propose here an alternative method which permits to shorten the computational domain and gives some criteria concerning the matching of inner and outer solutions on the surrounding surface.

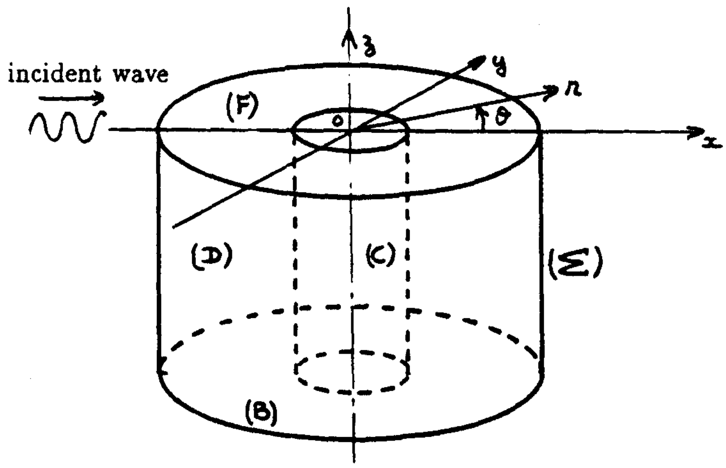


figure 1 : description of the domain

We recall the outline of the method. The basic idea results from considerations on the second-order diffraction potential $\phi_D^{(2)}$ at large radial distances, where two components may be identified : a locked component (ϕ_L) satisfying the non–homogeneous free–surface condition for which the two leading terms in $\frac{1}{\sqrt{kr}}$ and $\frac{1}{kr}$ can easily be derived, and a free component (ϕ_F) satisfying the homogeneous free surface condition.

Assuming this decomposition to hold on a fictitious cylinder (Σ) at a finite distance from the physical cylinder (C), a Sommerfeld type radiation condition for (ϕ_F) can be formulated as :

$$\phi_{F,r} - ik_{20}\phi_F = 0 \quad \text{at} \quad r = R_\Sigma \tag{1}$$

and an inner Boundary Value Problem (P) is derived :

$$(P)\begin{cases} \Delta\phi_D^{(2)} = 0 & \text{in the fluid } (D) & (2)\\ g\phi_{D,z}^{(2)} - 4\omega^2\phi_D^{(2)} = \alpha & \text{on the free–surface (F) } (z=0) & (3)\\ \phi_{D,r}^{(2)} = -\phi_{I,r}^{(2)} & \text{on the cylinder } (C)\ (r=a) & (4)\\ \phi_{D,z}^{(2)} = 0 & \text{on the bottom } (B)\ (z=-h) & (5)\\ (\phi_D^{(2)} - \phi_L)_{,r} = ik_{20}(\phi_D^{(2)} - \phi_L) & \text{on the fictitious cylinder } (\Sigma)\ (r = R_\Sigma) & (6)\end{cases}$$

α depends quadratically on the incident and diffracted potentials of the first–order; it can be expressed as follows :

$$\frac{\alpha}{a^2\omega^3} = i(1 - \frac{1}{2sh^2kh})\{D^2 + 2ID\} + \frac{i}{k^2th^2kh}\{2I_{,r}D_{,r} + \frac{2}{r^2}I_{,\theta}D_{,\theta} + D_{,r}^2 + \frac{1}{r^2}D_{,\theta}^2\} \tag{7}$$

where $I = e^{ikr\cos\theta}$ and D denotes the local complex amplitude of the first-order diffracted waves. $\phi_I^{(2)}$ is the solution of the second-order problem without any perturbation of the body. k_{20} is the solution of the linear dispersion relation for the double frequency : $4\omega^2 = gk_{20}thk_{20}h$.

This problem (P) is classically solved with the integral equation method; however we have found that a good modeling of the free-surface elevation requires a very refined discretization. Moreover, we are not able to control correctly the matching of the inner and outer solutions on the fictitious cylinder. Thus we focalized our attention on the decomposition of $\phi_D^{(2)}$ into ϕ_L and ϕ_F on (Σ).

Locked components : they are solutions of the equations (2),(3),(5) of the problem (P). α can be developed as : $\alpha = \alpha^{id} + \alpha^{dd}$ where :

$$\alpha^{id}(r,\theta) = \frac{e^{ikr(1+\cos\theta)}}{\sqrt{kr}} \cdot (\alpha_1^{id}(\theta) + \frac{\alpha_2^{id}(\theta)}{kr} + \frac{\alpha_3^{id}(\theta)}{(kr)^2} + ...) \tag{8}$$

$$\alpha^{dd}(r,\theta) = \frac{e^{2ikr}}{kr} \cdot (\alpha_1^{dd}(\theta) + \frac{\alpha_2^{dd}(\theta)}{kr} + \frac{\alpha_3^{dd}(\theta)}{(kr)^2} + ...) \tag{9}$$

This decomposition suggests to develop ϕ_L in the same way : $\phi_L = \phi_L^{id} + \phi_L^{dd}$ where :

$$\phi_L^{id} = \frac{e^{ikr(1+\cos\theta)}}{\sqrt{kr}} \cdot (f_1^{id}(\theta,z) + \frac{f_2^{id}(\theta,z)}{kr} + \frac{f_3^{id}(\theta,z)}{(kr)^2} + ...) \tag{10}$$

$$\phi_L^{dd} = \frac{e^{2ikr}}{kr} \cdot (f_1^{dd}(\theta,z) + \frac{f_2^{dd}(\theta,z)}{kr} + \frac{f_3^{dd}(\theta,z)}{(kr)^2} + ...) \tag{11}$$

by replacing successively ϕ_L^{id} and ϕ_L^{dd} in the Laplace equation, differential equations in z are obtained :

$$f_{j,zz}^{id} - 2k^2(1+\cos\theta)f_j^{id} = F^{id}(f_1^{id}, f_2^{id}, ..., f_{j-1}^{id}) \tag{12}$$

$$f_{j,zz}^{dd} - 4k^2 f_j^{dd} = F^{dd}(f_1^{dd}, f_2^{dd}, ..., f_{j-1}^{dd}) \tag{13}$$

which are solved as soon as the right hand sides F^{id} and F^{dd} are known, the integration constants being determined by the conditions (3) and (5). Their analytical calculation becomes rapidly tedious, and in our numerical application we truncate the summation at the term in $O(\frac{1}{(kr)^2})$, (i.e. up to the terms f_2^{id} and f_2^{dd}).

Free components : they are solutions of the equations (2),(3),(5) of the problem (P), but $\alpha = 0$ and we add the radiation condition (eq. 1). The solutions are well known. We can develop them on a basis of eigen-functions. We express them as follows :

$$\phi_F = \frac{chk_{20}(z+h)}{chk_{20}h} \sum_{m=0}^{+\infty} A_{m0}^{(2)} H_m(k_{20}r)\cos m\theta + \sum_{i=1}^{+\infty} \cos k_{2i}(z+h) \sum_{m=0}^{+\infty} A_{mi}^{(2)} K_m(k_{2i}r)\cos m\theta \tag{14}$$

and we note them :

$$\phi_F = \sum_{m=0}^{+\infty}\sum_{i=0}^{+\infty} A_{mi}^{(2)} \phi_{Fmi} \tag{15}$$

H_n and K_n are respectively the Hankel functions and modified Bessel functions. $(k_{2i})_{i=1,\infty}$ are the solutions of the relation : $-\frac{4\omega^2}{g} = k_{2i}tan(k_{2i}h)$.

Finally the Boundary Value Problem to be solved is nearly the same as (P) but the equation $\phi_{D,r}^{(2)} = \phi_{L,r} + \phi_{F,r}$ will be the new boundary condition on (Σ) which assumes only the matching of normal derivative of the potential. We will see later how we match the inner and outer potentials.

Integral equation for $\phi_D^{(2)}$: by replacing the Neumann condition of (P) in the following integral equation :

$$2\pi\phi - \int_{\partial D} \phi G_{,n} ds = -\int_{\partial D} G\phi_{,n} ds \tag{16}$$

where n denotes the inner normal, (∂D) the boundary of (D) and G the Green function (we add to G the symmetrical source with respect to the bottom so that $G_{,n} = 0$ on $z = -h$)
we obtain

$$2\pi\phi_D^{(2)} - \int_{(F)} \phi_D^{(2)}(G_{,n} + \frac{4\omega^2}{g} \cdot G)ds - \int_{(C)} \phi_D^{(2)} G_n ds - \int_{(\Sigma)} \phi_D^{(2)} G_{,n} ds =$$

$$\int_{(F)} \frac{\alpha}{g} \cdot G ds + \int_{(C)} \phi_{I,n}^{(2)} \cdot G ds - \int_{(\Sigma)} \phi_{L,n} \cdot G ds - \sum_{m=0}^{M_m-1} \sum_{i=0}^{M_i-1} A_{mi}^{(2)} \int_{(\Sigma)} \phi_{Fmi,n} \cdot G ds \qquad (17)$$

This equation is solved using a panel method with the unknown potential assumed to be constant over the surface of each panel. However the coefficients $A_{mi}^{(2)}$ are still unknown, so the right hand side of eq.(17) is decomposed in $M_m \cdot M_i + 1$ terms and we solve the linear system for each of them. It remains to match the inner and outer potentials on (Σ). $\phi_D^{(2)}$ and ϕ_F are developed in series with the same coefficients thus, by minimizing the following functional J :

$$J(M) = ||\phi_D^{(2)} - \phi_F - \phi_L||_{(M \epsilon \Sigma)} \qquad (18)$$

we obtain the coefficients $A_{mi}^{(2)}$: $||.||$ denotes a norm which depends on the numerical scheme; in our numerical application we used the Householder's least–square method.

Matching on the surrounding cylinder

As mentionned before, the choice of R_Σ is quite determinant for the matching of $\phi_D^{(2)}$ on (Σ); so we defined three criterions :

- matching of the right hand side (α) of the free–surface condition. We have to make sure that the asymptotical expansion $(\alpha^{asymp.})$ provided by eq.(8) and (9) (to the truncation orders retained for ϕ_L^{id} and ϕ_L^{dd}) is correct. Thus we define an error δ as follows :

$$\delta = \frac{||\alpha - \alpha_{asymp.}||}{max||\alpha||_{\theta \epsilon \{0,\pi\}}} \qquad (19)$$

  δ only depends on kr and the geometry (a, h), thus the choice of (δ, a, h, k) implies that of R_Σ.

- the truncated locked components do not exactly verify the continuity condition in the fluid. So we calculate the laplacian of ϕ_L on (Σ) – by Finite Difference Method for instance – and we check that :

$$\max\left\{ \left|\left| \frac{\Delta\phi_L^{id}(M)}{2k^2(1+\cos\theta)\phi_L^{id}} \right|\right| + \left|\left| \frac{\Delta\phi_L^{dd}(M)}{4k^2\phi_L^{dd}} \right|\right| \right\}_{M(r,\theta,z)\epsilon(\Sigma)} \leq \varepsilon \qquad (20)$$

  the value of ε gives an idea of the commited error concerning the expansion of locked components.

- the discretization of the free–surface must be refined enough to have about 6 panels per second–order wave length $(\lambda^{(2)} = \frac{2\pi}{k_{20}})$.

Results and comments

We can find three main advantages to this alternative method :

- better evaluation of the free waves.
- optimization of the position of the surrounding cylinder : higher order representation (in kr) of the locked waves and the use of modified Bessel functions permits to shorten the computational domain.
- numerical advantages :
  1) resolution of real linear systems since the Sommerfeld radiation condition does not appear explicitly in the integral equation anymore.
  2) we use the same discretization of (Σ) for the resolution of integral equations and for the least square minimization of J.
  3) extended visualization of the free–surface since we know analytically the outer solution.

References.

[1] Second-order deformation of the free surface elevation around a vertical cylinder. The Third International Workshop on Water waves and Floating Bodies. B. Molin L. Boudet (1988)

[2] Quelques réflexions sur la résolution du problème de diffraction au deuxième ordre. B.Molin. 1^{eres} Journées de l'Hydrodynamique de Nantes (1987)

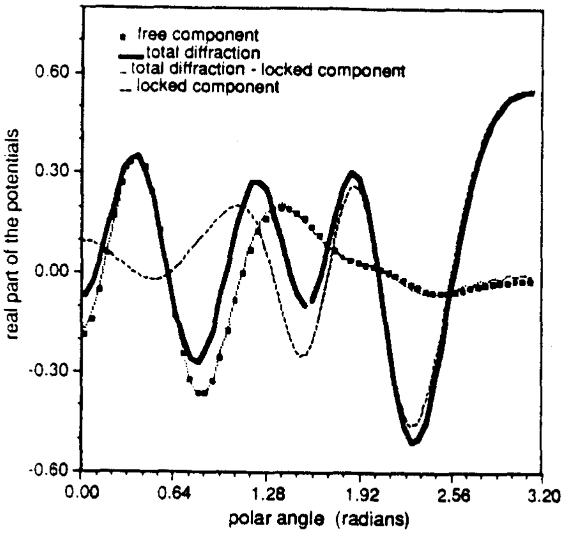


figure 2 : matching of the potentials (real part) at the intersection $(\Sigma)\bigcap(F)$

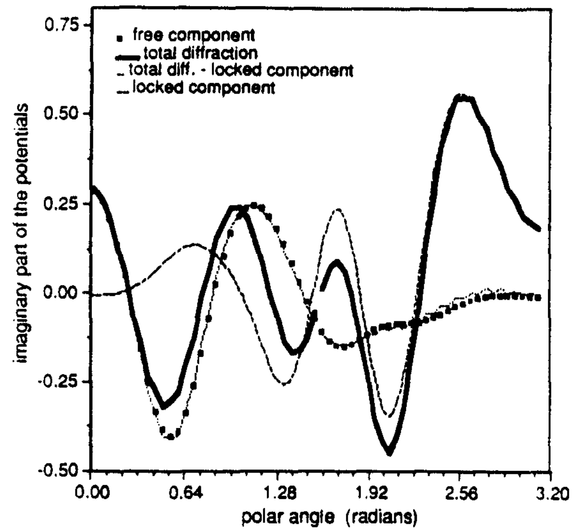
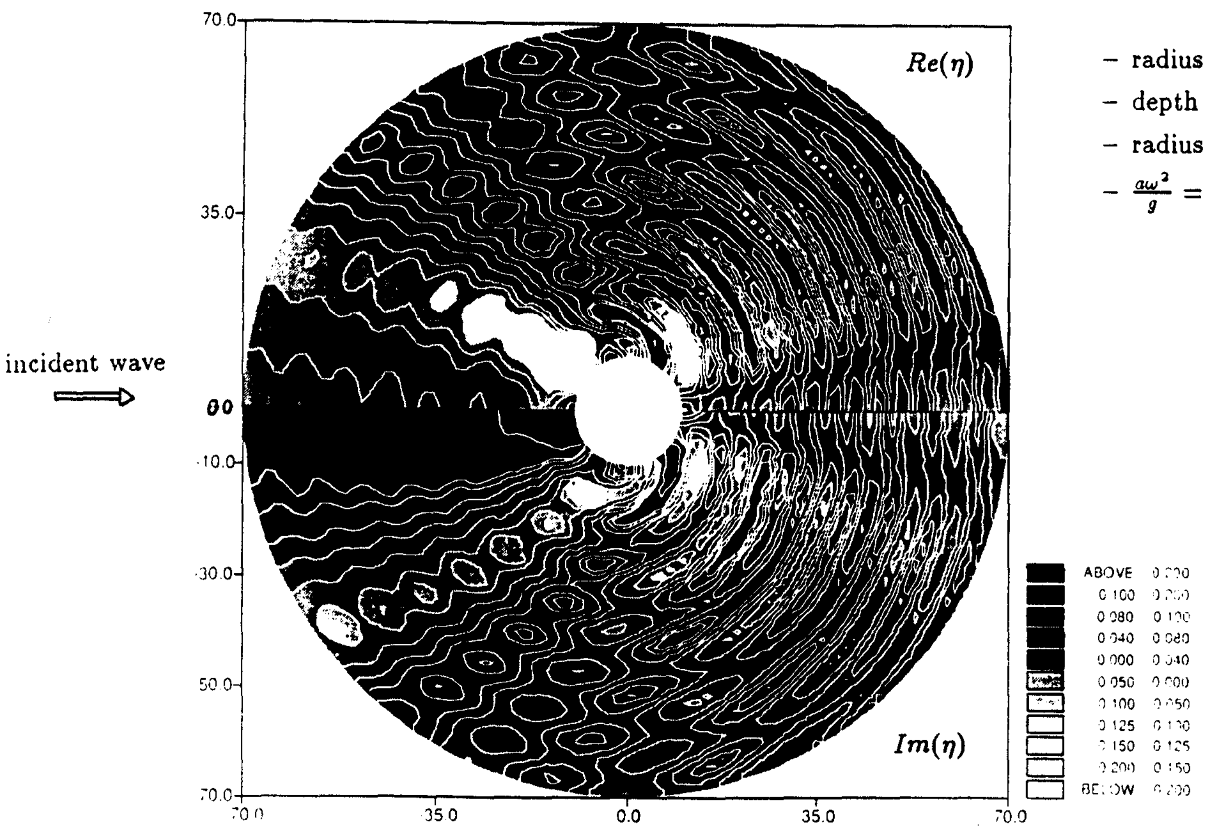


figure 3 : matching of the potentials (imaginary part) at the intersection $(\Sigma)\bigcap(F)$



- radius of (C) : $a = 10$
- depth $h = 10$
- radius of (Σ) $R_\Sigma = 35$
- $\frac{a\omega^2}{g} = 2$

figure 4 : free–surface elevation, contribution of $\phi_D^{(2)}$ for $r\epsilon[35,70]$ the elevation is calculated semi–analytically real and imaginary parts are represented

DISCUSSION

Ursell: Locked components: I agree that the imposed surface pressure for the second-order potential involves the factor $\exp\{ikr(1+\cos\theta)\}$ but does it follow that the potential itself involves this factor? There are no obvious harmonic functions involving such a factor, neither is it clear to me that geometrical interference will lead to parabolic wave crests. Question: Are the locked component terms really needed? Another question: Is there any solid evidence of the mathematical existence of a second-order potential satisfying the surface pressure and the radiation condition at infinity, or are the second-order waves reflected by the first-order waves?

Scolan & Molin: We agree that our reasoning is based more upon physical intuition and engineering needs than upon correct mathematics. We do not know what is a proper radiation condition for the second-order diffraction problem, and we leave that point to mathematicians!

We want to emphasize that the proposed expression of the locked potential is only valid asymptotically (for $R \to \infty$) and that it is only a component of the total second-order diffraction potential.

Yeung: The point was made that irregular frequencies could occur in such a method of solution. It was pointed out in my work of 1973 (Yeung, R.W., Rept NA-73-4, Univ. of California, Dept. of Naval Arch.) and also quite well-established since then the use of simple source formulation in a manner similar to my 1973 work is free of irregular frequencies since the standard Green function is not involved. I should point out that a similar method was developed by Shimada (J.Soc. of Naval Arch., 1986) for the 2nd-order diffraction problem. I want to commend the authors for a very careful piece of numerical work.

Scolan & Molin: 1) We agree with the discussor that, if a sommerfeld condition like (1) (and like is used in his thesis) is applied on the matching cylinder Σ, the homogeneous problem associated with (2)...(6) only admits the trivial solution.

In the method presented here the boundary condition (6) is replaced by a Neumann conditon. As a result, if the radius R_ε of the matching cylinder is such that

$$J_m'(k_{20}a)Y_m'(k_{20}R_\varepsilon)-Y_m'(k_{20}a)J_m'(k_{20}R_\varepsilon) = 0$$

for some value of m, the associated homogeneous problem admits non trivial solutions: the natural modes of the annular region inbetween the two cylinders. In such case the solution of the problem we solve is not unique and we should get into numerical trouble. We were surprised that no disagreement of this kind seemed to occur in our computations.

2) We know of the work by Shimada (1987). We do not agree with his improved expression of the locked potential.

Chau: Would you consider that the differences between your result with that given by Kim and Yue (h/a=4) near the free surface may due to the fact that you have only retained two evanescent modes in your expression of ϕ_F, since we know that more evanescent modes may be required at larger water depth.

Scolan & Molin: In fact the use of evanescent functions ($K_m(k_{2i}.R_\varepsilon)$), which decay exponentially in the radial direction is not completely justified since the argument $k_{2i}R_\varepsilon$ is very large; but no numerical problems occur because of the normalization of Bessel functions. A systematic study of the convergence of the results as a function of number of modified Bessel functions has been made. But we established that only a couple of them is necessary in the computation.

As for the difference which appears at the waterline (for the comparisons made with Kim and Yue's results), it is only due to the discretization.