SLENDER, STEEP AND BREAKING WATER WAVES

A. Anderson, D.A. Diver and D.H. Peregrine
School of Mathematics, University Walk, Bristol BS8 1TW, England

SUMMARY AND INTRODUCTION

We consider steady water waves in three dimensions,which may be considered to be slender in the same sense as "slender bodies",. That is, the wave shape is moving relative to the water in a direction which is nearly perpendicular to its maximum slopes. For example, slender waves are generated by rapidly moving ships and boats. We describe an initial study of waves alone with special emphasis on their mode of breaking. The work will soon be expanded to include moving bodies.

MATHEMATICAL DETAILS

The waves are taken to be moving with constant velocity U in the z direction, and with a "slenderness" expressed by supposing they have a typical long dimension of length L in the z direction with variations in the transverse x, y directions on a length scale b where

$$\frac{b}{L} = \epsilon \ll 1.$$

Their mathematical description is as steady waves on a current U. Dimensionless variables are introduced, with * denoting dimensional variables, as follows:

$$x^* = bx,\ y^* = by,\ z^* = Lz,\ \phi^* = (gb)^{\frac{1}{2}}b(Fz+\phi) \text{ and } \zeta^* = b\zeta$$

where ϕ^* is the velocity potential, $z^* = \zeta^*(x^*, y^*, t^*)$ is the free surface and $F = U/(gb)^{\frac{1}{2}}$ is a Froude number. With these variables Laplace's equation and its free surface boundary conditions become

$$\begin{aligned} \phi_{xx} + \phi_{yy} &= -\epsilon^2\phi_{zz} \\ F\epsilon\zeta_z + \phi_x\zeta_x - \phi_y &= -\epsilon^2\phi_z\zeta_z \\ F\epsilon\phi_z + \tfrac{1}{2}\phi_x^2 + \tfrac{1}{2}\phi_y^2 + \zeta &= -\tfrac{1}{2}\epsilon^2\phi_z^2. \end{aligned}$$

A change of variable to the pseudo-time: $t = z/\epsilon F$ and neglect of the $O(\epsilon^2)$ makes the above equations directly equivalent to those for the unsteady motion of two-dimensional water waves in the (x,y) plane. This permits computation of steep and overturning waves by boundary-integral methods such as the accurate and efficient program developed by Dold and Peregrine (1986) and its extensions. We note that in such programs the free surface is followed with a Lagrangian representation (X,Y) of surface points and

$$\frac{dX}{dt} = \phi_x,\ \frac{dY}{dt} = \phi_y,\ \frac{d\phi}{dt} = \tfrac{1}{2}(\phi_x^2 + \phi_y^2) - Y.$$

Pseudo-time t, and the true time, t*, are related along particle paths by

$$\left[\frac{b}{g}\right]^{\frac{1}{2}} \frac{dt}{dt^*} = 1 + \frac{1}{F^2}\phi_t,$$

which reinforces the implicit assumption that $\epsilon F = O(1)$ in order that the last term may be neglected.

BREAKING-WAVE EXAMPLES

Little is known about breaking, or limiting form of steady waves in three dimensions. Roberts (1983) gives some tantalizing results based on a Fourier series approximation. The "slender" case we illustrate correspond to linear two-dimensional standing waves. In figure 1 a selection of breaking examples is given to show the variety of cases that can occur with sinusoidal initial conditions as indicated. This variety and the range of behaviour in other breaking-wave examples has stimulated a closer study of the one feature that appears to be common to all breaking wave flows prior to the stage of jet formation. That is, there is a strong convergence of liquid to form the jet.

CONVERGENT FLOWS

Convergence is obtained by taking two opposing uniform flows. A smooth transition is obtained by initial conditions of a flat surface with a horizontal velocity component of

$$\phi_x = u_0(\tanh x/b_0)$$

The water depth is unity.

A wild variety of shapes occur, several examples are shown in figure 2. As may be seen, some relate to the standing wave examples and others develop as propagating waves. Details of these flows are under study and we find that the weakly three-dimensional interpretation of the flow field is very helpful. For example, the pressure field beneath these unsteady waves is vital to understanding their dynamics. Large pressure gradients, with a rapid increase in time occur. Gaining an understanding of these flows is difficult in two dimensions where Bernoulli's equation has a dominant ϕ_t term. Once this is translated into a ϕ_z term in the third dimension our greater familiarity with steady flows eases interpretation.

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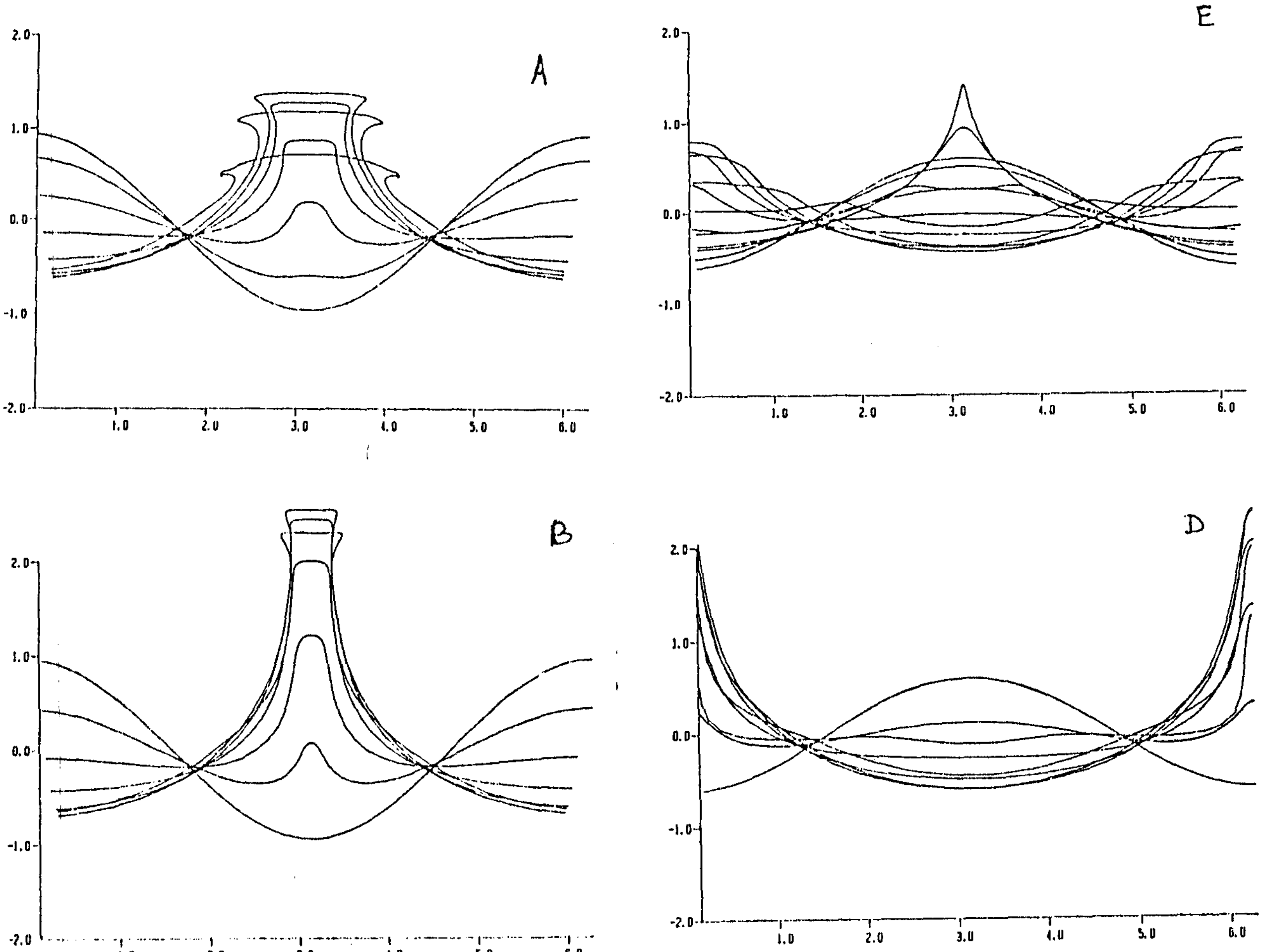
REFERENCES

J.W. Dold & D.H. Peregrine (1986) An efficient boundary-integral method for steep unsteady water waves. In "Numerical Methods for Fluid Dynamics II" Eds Morton & Baines, 671-679. Clarendon.

A.J. Roberts (1983) Highly nonlinear shortcrested water waves. J.F.M. 135, 301-321.



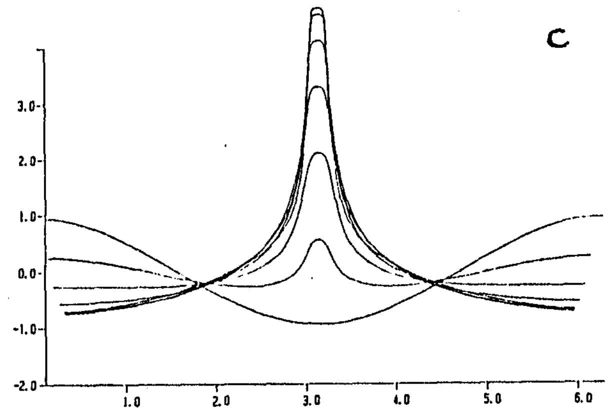
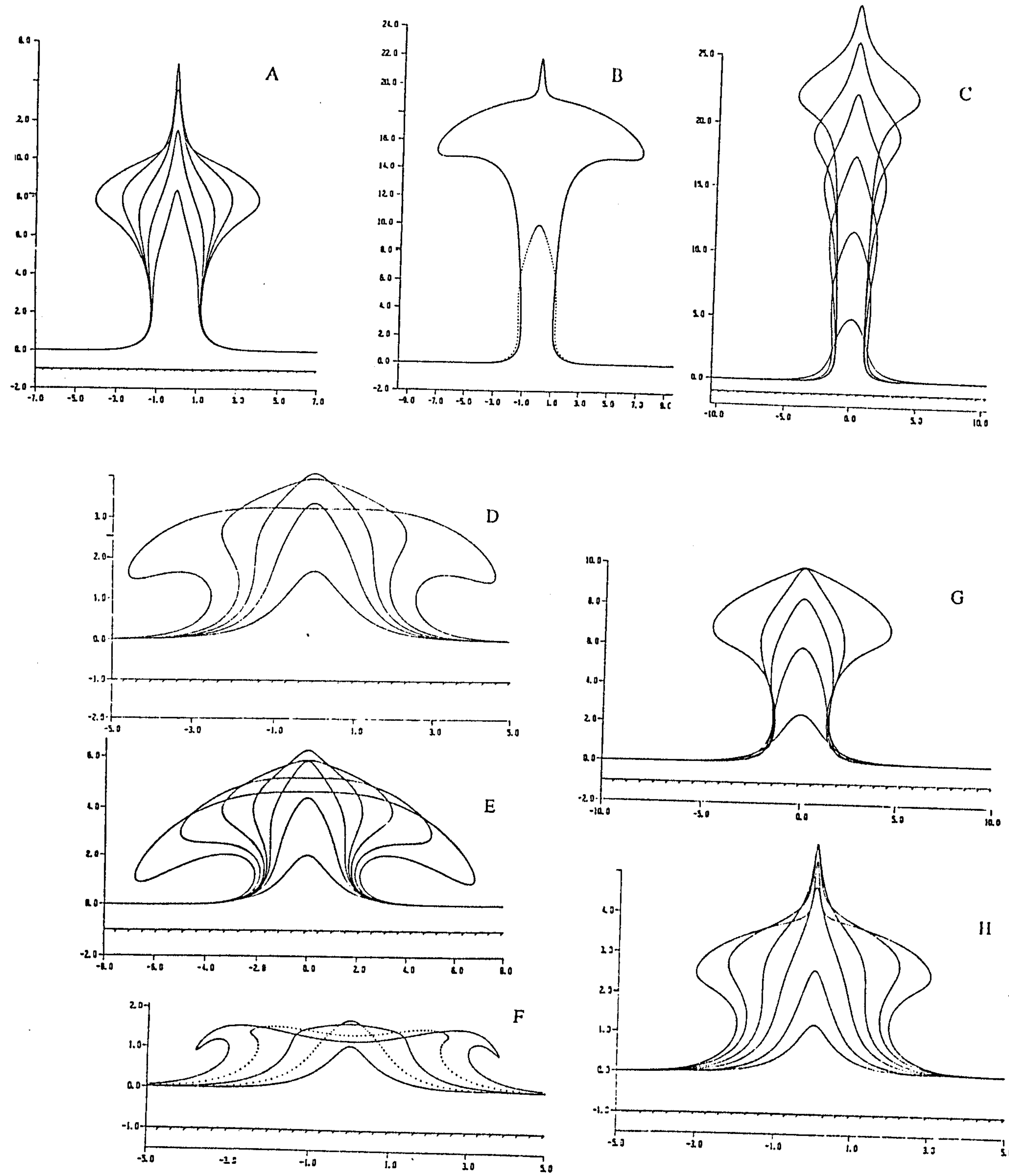


Figure 1

Key

$\zeta_0 = a_1 \cos(x), \quad \phi_0 = -a_2 \cos(x)$

Parameter values		
Fig	a_1	a_2
A	0.95	0.3
B	0.95	0.95
C	0.95	1.4
D	-0.6	0
E	-0.6	-0.9



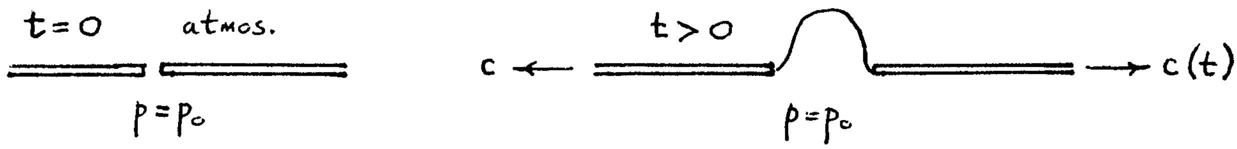
Key

Parameter values								
	A	B	C	D	E	F	G	H
u_0	4	6	8	2	2.8	1	4	2
b_0	2	4	6	2	2.8	1	4	1

Figure 2

DISCUSSION

Tulin: These results are so interesting that it would be very desirable to observe them experimentally. A suggestion: the case of a two-dimensional valve opening.



Anderson: Setting up the situation stated may be difficult: the flow here, which is initially stationary, may depend on $c(t)$. In the high-pressure cases, it would probably be adequate to pull the plates apart very quickly. For lower pressures, however, the effects of different plate movements may be significant. Setting up any experiment will have problems in keeping the more rapidly convergent flows smooth and free from violent splashing.

Tuck: If you want to study free surfaces behind a high-speed slender boat, why do your results show a *positive* free surface? Surely you should start with an *initial* profile involving a *depression* of the free surface?

Anderson: In my lecture, I showed some results from the convergent flow computations that are similar to the central part of the wakes forming behind a wide boat and a fast, thin boat or water-skier; see Fig. 3. The slender body theory states that $F^{-2}\phi_t$ must be small, and the flows in Fig. 3 have low enough ϕ_t to keep this term small when F is estimated for the 'real' flow situations suggested.

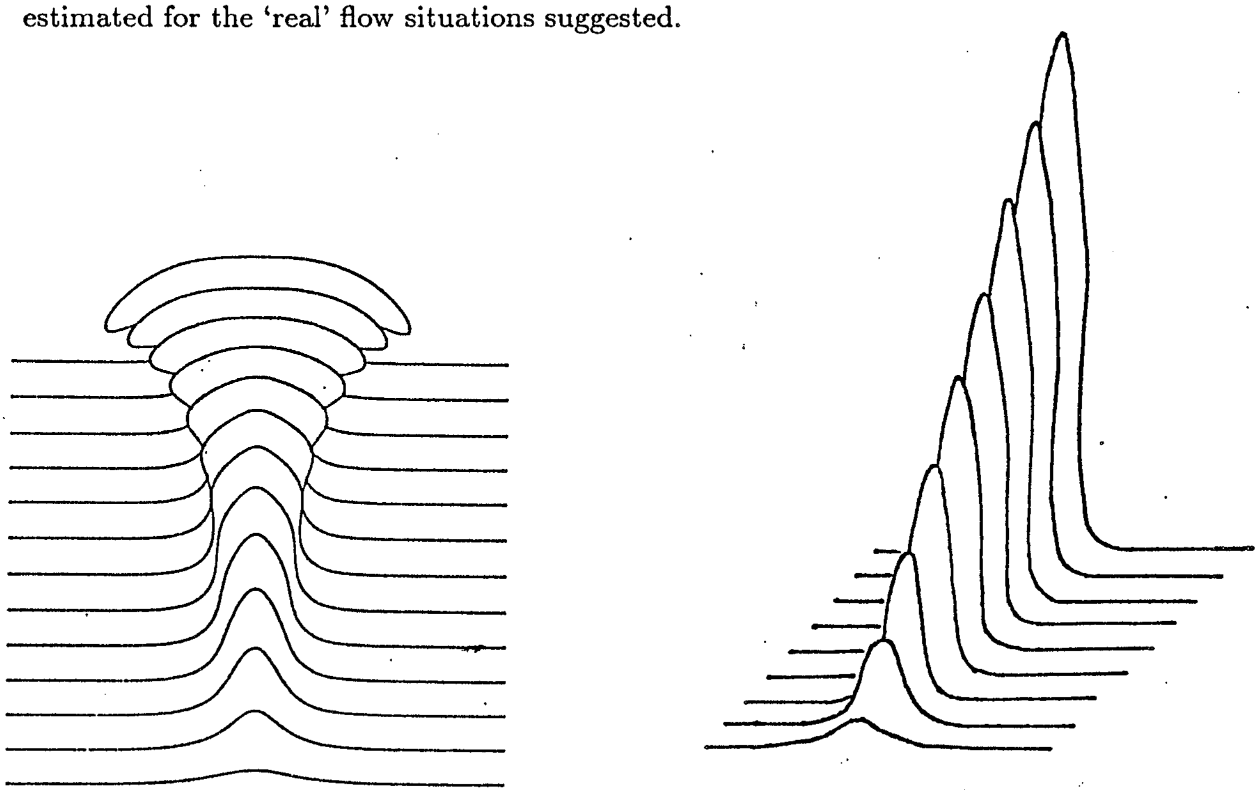
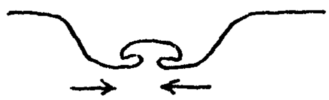
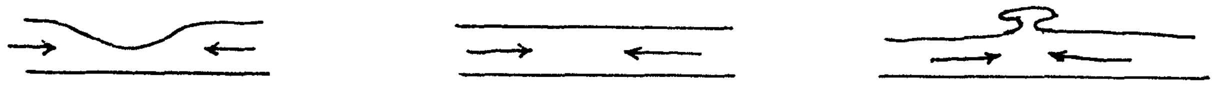


Figure 3. Two possible wake forms.

The curves shown can either be thought of as representing the flow at the bottom of the depression left by a boat (see right), or possibly as the flow after such a depression has closed, in which case it would be above the free surface (as calculated):





Grue: How does surface tension modify the pictures of your flow computations? Can the effect of surface tension be included in your model?

Anderson: Although the inclusion of surface tension seems straightforward, we understand that problems have been encountered by researchers using other boundary-integral programs. We can guess that surface tension will probably suppress the production of spikes, and circularise the 'bulbs' seen in the faster, small-scale flows. In the axisymmetric case, the bulb in free fall will become a drop of water and may detach, as is often seen after a drop has hit the surface of calm water which then splashes back up in a jet.

King: In your two-dimensional calculations, did you find any waves on the free surface?

Anderson: For slower flows, with $U \lesssim 2$, the flow develops into a bore moving outwards; this bore will catch any gravity waves moving outwards from the convergence at the centre. Faster flows will probably become a quasi-steady flow as the inflow continues. This flow will rise up and then overturn like a fountain: