MATHEMATICAL AND NUMERICAL STUDY OF TRAPPING WAVES

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It is well known that *trapping waves* can propagate along a straight coast over a protrusion of the sea-bed. Except for the case of a plane sloping beach (cf. STOKES [1], URSELL [2]), these waves cannot be calculated analytically. In the general case, JONES [3] proved that the study of trapping waves consists in an *eigenvalue problem* for an unbounded self-adjoint operator, the time-frequency appearing as a parameter. So he established the existence of trapping waves over any cylindrical submerged body, the depth beeing supposed infinite. Using similar techniques, GARIPOV [4], GRIMSHAW [5] and URSELL [6] proved existence results for the fundamental mode, in particular when the depth remains finite and becomes constant far enough from the coast. In the present work, their study is extended and various complementary results are established, concerning in particular the existence of modes, other than the fundamental one. Moreover, a numerical technique is presented, based on the *localized finite element method* (cf. LENOIR [7]), to compute trapping waves for arbitrary sea-bed profiles.

The model

Consider the irrotational motion of an inviscid incompressible fluid, and denote by $\Phi(x,y,z,t)$ the velocity potential, where x is the offshore direction, z the direction parallel to the coast-line and y the vertical coordinate. By the linear theory, the free surface boundary condition can be written on the mean free surface $\tilde{\Gamma}_F$ whose equation is $y = 0$. If the bottom $\tilde{\Gamma}_B$ has for equation $y = -h(x,z)$, the velocity potential is defined in the domain $\tilde{\Omega} = \{(x,y,z);\ x \geqslant 0,\ -h(x,z) \leqslant y \leqslant 0\}$, and must satisfy the following equations :

$$(1) \qquad \Delta\Phi = 0 \text{ in } \tilde{\Omega}\ ,\ \frac{\partial\Phi}{\partial n} = 0 \text{ on } \tilde{\Gamma}_B \text{ and } g\frac{\partial\Phi}{\partial n} + \frac{\partial^2\Phi}{\partial t^2} = 0 \text{ on } \tilde{\Gamma}_F,$$

where g denotes the acceleration of gravity.

We suppose that the sea-bed topography depends only of the distance x to the coast-line, i.e. $h(x,z) = h(x)$, and that the depth h takes a constant value h_∞ for x large enough.

The *trapping modes* (or *guided modes*) are particular solutions of problem (1) of the form $\boxed{\Phi(x,y,z,t) = \text{Re}\ (\varphi(x,z)\ e^{i(\omega t-\beta y)})}$, where ω and β are real and φ is square-integrable. For Φ to be solution of (1), φ must satisfy :

$$(2) \qquad \Delta\varphi = \beta^2\varphi \text{ in } \Omega\ ,\ \frac{\partial\varphi}{\partial n} = 0 \text{ on } \Gamma_B\ ,\ g\frac{\partial\varphi}{\partial n} = \omega^2\varphi \text{ on } \Gamma_F\ ,$$

where Ω (resp. Γ_B, Γ_F) denotes the intersection of $\tilde{\Omega}$ (resp. $\tilde{\Gamma}_B$, $\tilde{\Gamma}_F$) with the transverse plane z = 0. In fact, every square-integrable solution φ of (2) decays exponentially as x tends to infinity.

For any given value of the pulsation ω, the problem is to determine the discrete values of the propagation constant β such that problem (2) has non-trivial square integrable solutions φ. That is a *two-dimensional eigenvalue problem* of the form $C_\omega \varphi = \beta^2 \varphi$, where C_ω is a selfadjoint non-compact operator of $L^2(\Omega)$ depending on the parameter ω, β^2 is an eigenvalue of C_ω and φ an associated eigenfunction.

More generally, our aim is to describe the *dispersion curves* $\omega \longrightarrow \beta^2(\omega)$.

Previous results for particular topographies

As far back as 1846, STOKES [1] studied the case of a plane sloping beach, i.e. $h(x) = (\text{tg } \alpha)\, x$, with $\alpha < \pi/2$. He proved the existence of a guided mode for every positive value of the pulsation ω and found an explicit expression of the potential. Then URSELL [2] proved that the STOKES edge wave is in fact the first term of a sequence of $N(\alpha)$ guided modes, where $N(\alpha)$ is a finite number, independent of ω, which is a monotone decreasing function of α.

Then, various authors were concerned with the case of a shelf :

$$h(x) = h \text{ if } x < a \text{ and } h(x) = h_\infty \text{ if } x > a, \text{ with } h < h_\infty.$$

In particular, according with theoretical works of JONES [3], EVANS and McIVER [8] checked numerically that there is only one guided mode at low or high frequency. If the area of the shelf is large enough, other modes appear at intermediate frequencies.

Also the existence of guided modes above a submerged cylindrical cylinder was studied by URSELL [9] and EVANS & McIVER [10], using a Bessel expansion of the potential.

Mathematical analysis for arbitrary sea-bed profiles

GARIPOV (cf. LAURENTIEV & CHABAT [4]) handled the case of an arbitrary sea-bed profile h by using the spectral theory of self-adjoint non-compact operators. By the *Min-Max Principle* (cf. REED & SIMON [11]), he proved that the existence of guided modes is equivalent to the existence of some potential φ such that :

$$\int_\Omega |\nabla\varphi|^2 dxdy - \omega^2 \int_{\Gamma_F} \varphi^2 dx < - \beta_\infty^2(\omega) \int_\Omega \varphi^2 dxdy ,$$

where $\beta_\infty(\omega)$ is the solution of the classical dispersion relation :

$$\beta \ \text{th}(\beta h_\infty) = \omega^2 .$$

Using an appropriate test-function φ, GARIPOV established the existence of trapping waves at high frequency. We complete this result by proving that the fundamental mode exists at every positive frequency if and only if :

$$\int_0^{+\infty} (h(x) - h_\infty)\, dx \leqslant 0 .$$

Moreover, we prove, for an arbitrary sea-bed profile h, that there is at most one guided mode at low frequency.

Using again the Min-Max Principle, we study conditions for the existence of modes, other than the fundamental one. The results vary according to whether we consider a *sea-cliff* ($h'(0) = +\infty$) or a *sloping shore* ($h'(0) = tg\ \alpha$, $\alpha < \frac{\pi}{2}$) .

The results for the cliff generalize the ones for the shell : indeed, we prove that there is at most one guided mode at high frequency.

On the other hand, for a sloping shore, there are at least $N(\alpha)$ guided modes at high frequency, where $N(\alpha)$ is the number of guided modes for the plane sloping beach with angle α. We establish partially the converse part for large angles ($\alpha \geqslant \pi/4$).

Eventually, we consider a beach, tangent to the free surface. In that case, ve prove that the number of guided modes increases indefinitly with the frequency.

Numerical method

The problem is set in the whole fluid domain, which is unbounded. However, it is possible to mesh only the domain of interest which is located near the shore, by using the *Localized Finite Element Method* (cf. LENOIR [7]).

Consider a domain $\hat{D}$, obtained by truncating the whole domain D by a vertical boundary $\hat{\Sigma}$ with equation $x = \sigma$. The value of σ is supposed to be large enough so that h is constant for x greater than σ (cf. FIGURE 1).

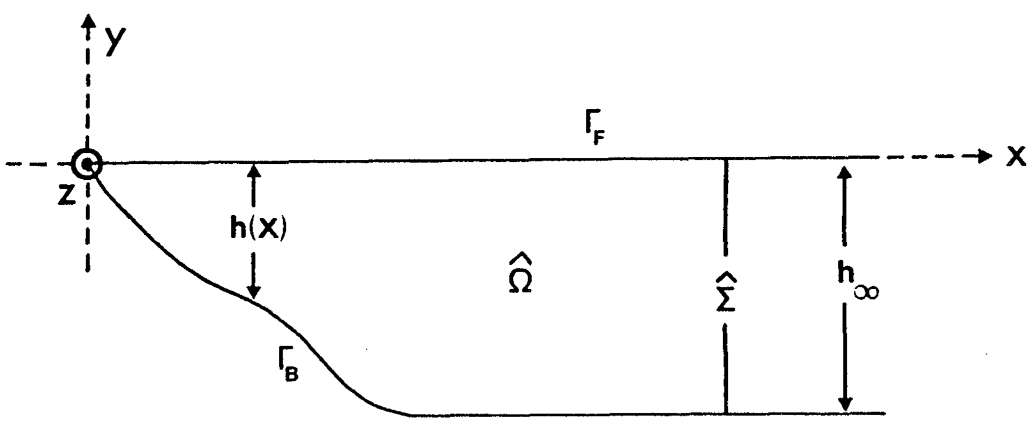

FIGURE 1

Then, one can prove that the potential satisfies on $\hat{\Sigma}$ a boundary condition of the form

$$\frac{\partial\varphi}{\partial n} = T_\sigma \varphi$$

whose explicit expression can be given using an analytic expansion of the potential outside $\hat{\Omega}$.

In fact, the operator T_σ depends non-linearly from the eigenvalue β^2. So the initial problem reduced to a non-linear eigenvalue problem set in the bounded domain $\hat{\Omega}$.

To solve it, we use a fixed point technique whih is proved to converge ; at each step, eigenvalues are computed by the inverse power method.

For the numerical approximation, we use finite elements and the previous expansion is truncated.

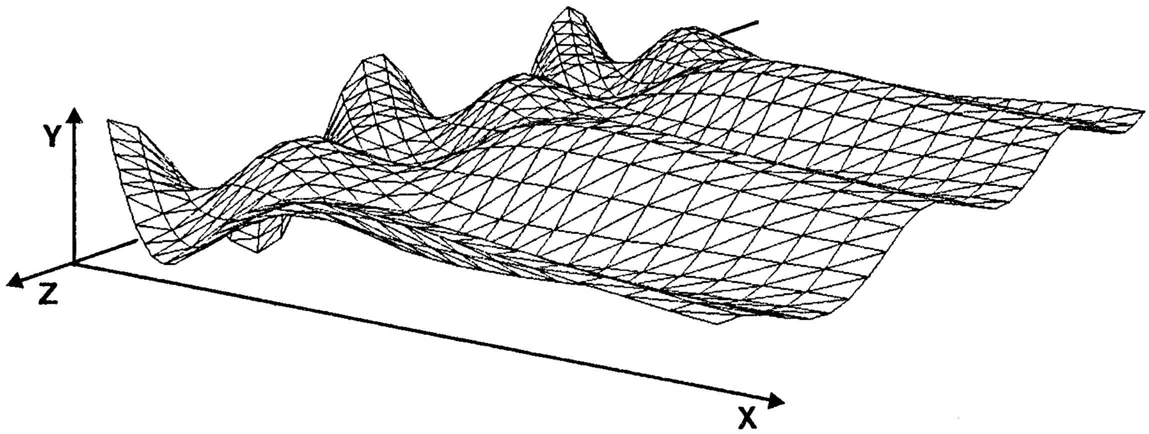

FIGURE 2 : Free-surface elevation. Third mode for a sloping shore.

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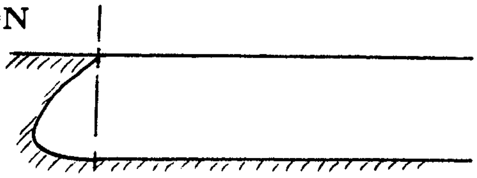
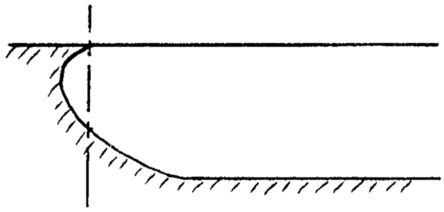
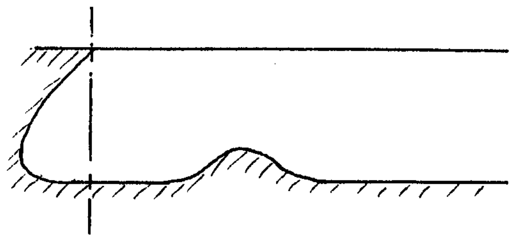
DISCUSSION

Tuck: Are there trapping modes for 'overhanging' cliffs? Here, I mean beaches of angle $> \frac{\pi}{2}$, as shown on the right:



Bonnet: The part of the coast which is located 'at the left' of the coastline does not influence most of our results. For instance, if the part of the bottom located directly below the free surface is flat (as in your sketch), there is no trapping mode. On the other hand, in the cases sketched below, trapping modes do exist, at least at higher frequencies.





McIver: Have you made any calculations for a circular cylinder, with its centre below the mean free surface, that intersects the free surface? For high frequencies, this is much like a plane beach, but what happens at lower frequencies?

Bonnet: We have not yet made any numerical computations for this case. Nevertheless, our numerical method is adapted for the treatment of this problem and we intend to make some tests soon.

At lower frequencies, we can only state that the fundamental mode exists.

Ursell: Two references: there is a shallow-water theory due to Carl Eckart ⟦1⟧; I would also like to draw your attention to a recent paper by Aranha ⟦2⟧. For a submerged body, can you prove that the number of trapping modes $\to \infty$ as the depth of submersion $\to 0$?

Bonnet: The answer to your question is not contained in our results. We can just say that, when the submerged body is tangent to the free surface, there exists an infinite number of trapping modes. To obtain continuity results of the spectrum of the operator with respect to the position of the submerged body would require a deeper analysis, but we think that our techniques should allow us to progress in that direction.

Peregrine: A different approach to edge-wave problems that you may not be aware of is that of R. Smith ⟦3⟧. He considered edge-wave propagation along a slowly varying coast.

Bonnet: Thank you. Smith's work appears as a continuation of the subject we have treated and not really as an alternative approach.

References

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⟦3⟧ R. Smith, 'Asymptotic solutions for high-frequency trapped wave propagation', *Phil. Trans. Roy. Soc. A***268** (1970) 289–324.