A COMMENT ON THE SECOND ORDER DIFFRACTION POTENTIAL FOR A VERTICAL CYLINDER

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Introduction

At the third workshop we presented results for the second order diffraction potential, and discussed certain aspects of numerical methods which we were investigating for arbitrary three dimensional bodies. These have now been developed further[1,2] and applied in the assessment of TLP springing. To verify the methodology we have also developed a series solution for a vertical cylinder. At first sight this solution does not appear to satisfy the inhomogeneous second order free surface boundary condition, and the purpose of this note is to reconcile this apparent conflict.

Formulation

We designate the second order incident and scattered potentials as $\phi_I^{(2)}$ and $\phi_S^{(2)}$ respectively. Our aim is to evaluate $\phi_S^{(2)}$ for a vertical circular cylinder of radius a standing in water of depth d subject to regular waves of frequency ω. We use polar coordinates (r,θ,z) with the origin where the axis of the cylinder intersects the plane of the mean free surface, and the z axis pointing vertically upwards. The free surface condition satisfied by $\phi_S^{(2)}$ is of the form

$$\frac{\partial \phi_S^{(2)}}{\partial z} - \frac{4\omega^2}{g}\phi_S^{(2)} = F(r,\theta)$$

$$= \sum_{m=0}^{\infty} \epsilon_m F_m(r)\cos_m\theta \qquad (1)$$

The solution for $\phi_S^{(2)}$ may be expressed as a mixed distribution of sources and dipoles over the body surface S_B, together with an integral over the free surface S_F. In terms of a linear wave source Green function $G(\vec{x},\vec{x}_0)$ pulsating at frequency 2ω, the general form can be abbreviated as

$$\phi_S^{(2)} = -\iint_{S_B}[\phi_S^{(2)}\frac{\partial G}{\partial n} + G\frac{\partial \phi_I^{(2)}}{\partial n}]dS - \iint_{S_F} FGdS \qquad (2)$$

In the case of the circular cylinder, it is possible to derive a Green function G_B which, in addition to satisfying the homogeneous free surface condition, and seabed and radiation conditions, also satisfies the condition of no flow through the surface of the cylinder[3]. We have then the explicit solution

$$\phi_S^{(2)} = -\underbrace{\iint_{S_B} G_B \frac{\partial \phi_I^{(2)}}{\partial n} dS}_{I} - \underbrace{\iint_{S_F} F G_B dS}_{II} \tag{3}$$

The term designated I corresponds to free wave contributions satisfying a homogeneous free surface boundary condition. It is the second term which concerns us here, which we shall call $\phi_{SL}^{(2)}$. This corresponds to a locked wave component, which is required to satisfy (1).

The appropriate Green function is

$$G_B = \frac{-1}{2\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} H_{mn}(r_0, r)\ Z_n(\kappa_n z_0) Z_n(\kappa_n z) \epsilon_m \cos m\theta_0 \cos m\theta \tag{4}$$

where

$$H_{mn}(r_0, r) = -\left[\frac{I_m'(\kappa_n a)}{K_m'(\kappa_n a)} K_m(\kappa_n r_<) - I_m(\kappa_n r_<)\right] K_m(\kappa_n r_>) \tag{5}$$

$$Z_n(\kappa_n z) = \left[\frac{4\kappa_n}{\sin 2\kappa_n d + 2\kappa_n d}\right]^{\frac{1}{2}} \cos \kappa_n (z+d) \tag{6}$$

$$r_> = \max(r_0, r), \qquad r_< = \min(r_0, r)$$

and κ_n are roots of $\kappa_n \tan \kappa_n d = 4\omega^2/g$. Hence for the circular cylinder we obtain

$$\phi_{SL}^{(2)}(r_0, \theta_0, z_0) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P_{mn}(r_0) Z_n(\kappa_n z_0) Z_n(0) \epsilon_m \cos m\theta_0 \tag{7}$$

where

$$P_{mn}(r_0) = \frac{1}{2\pi}\int_0^{2\pi}\int_a^{\infty} H_{mn}(r_0,r)\ F(r,\theta)r dr\ \cos m\theta d\theta$$

$$= \int_a^{\infty} H_{mn}(r_0,r)\ F_m(r) r dr \tag{8}$$

Because each term Z_n in the series (7) satisfies the homogeneous free surface boundary condition it is not obvious that our expression for $\phi_{SL}(2)$ satisfies (1). We therefore have to examine the problem more carefully. Substituting (8) into the left hand side of (1) we have

$$\frac{\partial \phi_{SL}^{(2)}}{\partial z} - \frac{4\omega^2}{g}\phi_{SL}^{(2)} = \sum_{m=0}^{\infty} \epsilon_m \cos m\theta_0 \sum_{n=0}^{\infty} P_{mn}(r_0)\frac{4\kappa_n^2 \sin(-\kappa_n z_0)}{\sin 2\kappa_n d + 2\kappa_n d}$$

$$= \sum_{m=0}^{\infty} \epsilon_m \cos m\theta_0 \left[\sum_{n=0}^{N} S_{mn} + \sum_{n=N+1}^{\infty} S_{mn}\right] \tag{9}$$

where

$$S_{mn} = P_{mn}(r_0)\frac{4\kappa_n^2 \sin(-\kappa_n z_0)}{\sin 2\kappa_n d + 2\kappa_n d} \tag{10}$$

We can now examine the asymptotic behaviour of the terms S_{mn} for large n. Integrating (8) by parts we find that as $n \to \infty$

$$P_{mn}(r_0) \to \frac{F_m(r_0)}{\kappa_n^2} .$$

Since also then $\kappa_n d \to n\pi$, we obtain

$$S_{mn} \to \frac{2}{\pi} F_m(r_0)\frac{\sin(-n\pi z_0/d)}{n}$$

Hence from Jolley[4]

$$\sum_{n=N+1}^{\infty} S_{mn} = \frac{2}{\pi} F_m(r_0) \sum_{n=N+1}^{\infty} \frac{\sin(-n\pi z_0/d)}{n} = F_m(r_0)(1+z_0/d) \tag{11}$$

We thus discover that in the limit as $z_0 \rightarrow 0$, the right hand side of (9) does indeed tend to $F(r,\theta)$ as required by (1).

References

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