ABSORBING BOUNDARY CONDITIONS FOR LINEAR GRAVITY WAVES.

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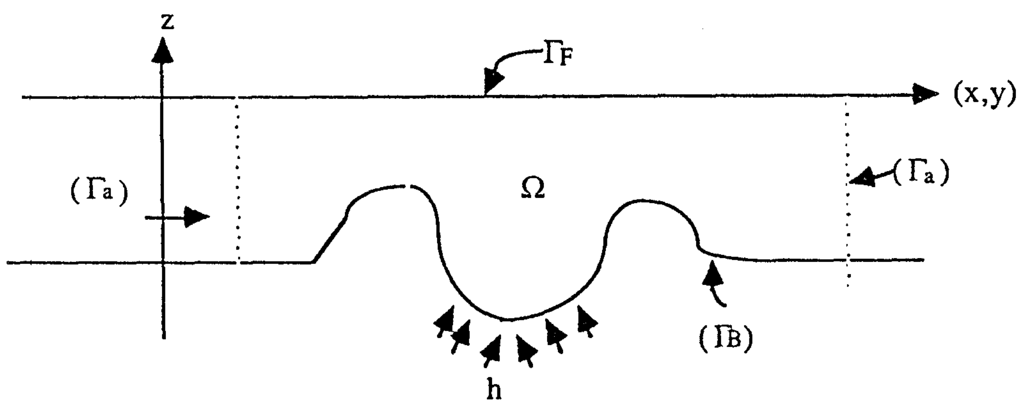
1) Introduction , position of the problem

The propagation of water waves in the linear mode is governed by the following system of equations :

$$(1.1)\quad \begin{cases} \Delta \phi = 0 & \text{in } \Omega \subset \mathbb{R}^3 \\ \dfrac{1}{g}\dfrac{\partial^2 \phi}{\partial t^2} + \dfrac{\partial \phi}{\partial z} = 0 & \text{on } \Gamma_F \\ \dfrac{\partial \phi}{\partial n} = h & \text{on } \Gamma_B \end{cases}$$



Where Ω denotes the region occuped by the fluid, Γ_F is the free surface and Γ_B is the bottom of the ocean .
The function ϕ represents the velocity potential : at any point (x,y,z) in Ω, the velocity of the fluid is given by :

$$v(x,y,z) = - \nabla \phi\, (x,y,z,t)$$

The function h is the source term.

We shall assume for our purpose that it compactly supported. The linearized mathematical model (1.1) is valid for the study of the irrotationnal flow of a perfect fluid as long as one is interested in weak amplitude waves. In this case, the free surface is taken to coïncide with the plane $z = 0$ (see figure 1.1) and the shape of the wave at time t is given by the surface :

$$(1.2) \quad z = \eta(x,y,t) = \frac{1}{g}\frac{\partial\phi}{\partial t}(x,y,o,t)$$

In many practical situations the domain Ω is unbounded in the horizontal directions x and y and has a constant depth L outside a bounded region, i.e. for $x^2 + y^2 > R^2$. Of course to be able to perform numerical calculations, one is led to bound artificially the domain of computation. This truncature procedure naturally poses the problem of the boundary conditions to be prescribed on the artificial boundary (Γ_a). We propose here a family of absorbing (or non reflecting) boundary conditions for both the 2D and the 3D case. This solution should be an efficient alternative to a completely different approach proposed previously by LENOIRE and VERRIERE [5] and is clearly related to the artificial boundary conditions currently used for other mathematical models of wave propagation ([1],[2]).

The absorbing boundary conditions

Before going into the details of the construction of absorbing boundary conditions, let us recall the main properties of the mathematical model (1.1). Although it does not correspond to a strictly hyperbolic problem, it has many properties analogous to the standard model of acoustic wave propagation.

In particular, the energy associated to ϕ :

$$(2.1) \quad E(\phi,t) = \frac{1}{2}\int_\Omega |\nabla\phi|^2 dx + \frac{1}{2g}\int_{\Gamma_F}\left|\frac{\partial\phi}{\partial t}\right|^2 d\sigma$$

is conservered as soon as the source term is equal to 0. Moreover the physical phenomenon involved is horizontal wave propagation, the vertical component only influences the amplitude of the wave. The variations of the propagation velocity, and therefore reflexion, transmission and diffraction phenomena are due to the variations of the depth.

Neverthless it is important to emphasize some differences between (1.1) and classical 3D wave propagation models :

- waves described by (1) propagate in fact with an infinite velocity, due to the Laplace equation for ϕ
- The propagation concerns only 2 space directions (and even 1 space direction in dimension 2)
- In the case of a constant depth, harmonic plane waves are dispersive and obey the following dispersion relation :

$$\omega^2 = g|k|\,th(|k|L)$$

where ω is the pulsation and $k = (k_x, k_y)$ is the wave vector.

For the construction of appropriate non reflecting boundary conditions, all these specific properties, and in particular the dispersion phenomenon, have to be taken into account. The method we follow for the construction is now rather classical :

- we construct,using the fact that the depth is constant in the exterior region, an exact transparent boundary condition (i.e. a condition for which the solution in the bounded domain coincides with the restriction of the solution in the unbouned domain).

- we approximate this boundary condition with the help of purely local operators, namely differential operators.

This step of approximation is of course the most important one and must be carried out in such away as to satisfy the following criteria :

- the approximate problem in the bounded domain must be mathematically well-posed (*)

- the numerical approximation of this new problem must be easy and cheap. This is a practical criterion.

- the parasitic reflections due to the presence of the artificial boundary must be minimized. This is a precision criterion.

The solution we propose differs according to whether one considers 2D or 3D problems :

i) In the 2D case, the approximation consists of two phases :

- we first get a local approximation in space by discarding the evanescent waves.
- we achieve the completly local approximation in space and time via an approximation of the dispersion relation. More precisely if $|K| = \omega f(\omega)$ denotes the solution of equation (2.2), we replace the function $f(\omega)$ by a rational function (if $g = L = 1$) :

$$(2.3) \quad f(\omega) \quad f_n(\omega) = 1 - \sum_{k=1}^{n} \frac{b_k \omega^2}{1 - a_k \omega^2}$$

where a_k and b_k are real numbers which must satisfy :

$$(2.4) \quad 1 - \sum_{k=1}^{n} \frac{b_k}{a_k} > 0$$

The resulting absorbing boundary condition involves auxiliary unknown functions defined on (Γ_a). It can be written

$$(2.5) \quad \begin{cases} \dfrac{\partial \phi}{\partial n} + \dfrac{\partial \phi}{\partial t} - \displaystyle\sum_{k=1}^{n} \dfrac{\partial \psi_k}{\partial t} = 0 \\ a_k \dfrac{\partial^2 \psi_k}{\partial t^2} + \psi_k = \phi \end{cases}$$

*) This is a stability criterion .

ii) In the 3D case, the approximation procedure requires a third phase to take into account the horizontal direction tangential to the boundary. Physically this corresponds to the fact that we have to deal with waves reaching the artificial boundary with different angles of incidence. In order to treat this difficulty, we adapt the classical method developed by ENGQUIST-MAJDA [2] for acoustic waves.
The outline of our talk will be the following one :

1°) Heuristic presentation of the construction of the approximate artificial boundary condition

2°) A short mathematical analysis of the wellposedness of this problem (we had to develop a generalization of the well-known theory of KREISS[4] for parabolic systems - (2.4) appears in this framework as a stability condition [3])

3°) A presentation of both 2D and 3D numerical results validating our approach.

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