Second order diffraction forces on a submerged body by second order Green function method.

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The determination of second order forces on floating or submerged bodies is an important problem for offshore applications. The second order problem is more complicated than the linear one due to the infinite extent of the non-homogenous free surface condition.
Previously second order problems have been solved using various integral equation methods.These methods require time consuming integration over the free surface and depend on a highly accurate first order solution.
A different approach which avoids integration over the free surface is developed by Sclavounos (1988).Studying two incoming waves,Fourier transform is used to derive two fundamental second order Green functions,called the Diffracion and the Radiation Green function respectivly. The Diffraction Green function corresponds to the second order interaction of an incoming linear wave with a submerged point source and the radiation Green function represents the second order interaction of two submerged point sources at different location and of different frequencies.
Explicitt particular solutions ϕ_p that together satisfies the second order free surface condition are obtained from the second order Green functions.Corresponding homogenous potentials ϕ_h subject to a linear free surface condition are required to ensure that the potentials $\phi_p + \phi_h$ satisfie the homogenous body boundary condition.
In this work we have studied the simple problem of a submerged restrained body in 2D,in order to investigate the method. The circular contour has been chosen as a numerical example,and the solution is compared to a previous solution by Vada(1987) using an integral equation method.

1 The second order forces

The particular potentials ϕ_p are obtained by applying the Fourier transform approach proposed by Sclavounos.As a starting point we have to use the linear wave-source Green function in 2D expressed as the Fourier integral

$$G(\vec{x},\vec{\xi}) = -\frac{1}{2}\int_{-\infty}^{\infty} du\left(\frac{e^{|u|(z+\zeta)}(|u|+\nu)}{|u|(|u|-\nu^{\epsilon})} + \frac{e^{-|u||z-\zeta|}}{|u|}\right)e^{iu(x-\xi)} \tag{1}$$

Here $\nu^\epsilon = \nu - i\epsilon$ indicates the integration path around the pole,using residue calculation. ϵ is a small positive parameter,$\vec{x} = (x, z)$ is the field point and $\vec{\xi} = (\xi, \zeta)$ is the source point. $\nu = \omega^2/g$ is the wavenumber,ω is the frequency and g the accelration due to gravity.
The total second order force (and moment) is found by integrating the pressure around the body contour,utilizing the Bernoulli equation. For simplicity only the contribution from the second order potential in the Bernoulli equation is presented below.
In the force expressions the potential ϕ_h is eliminated introducing an auxiliary linear radiation potential ψ_i with frequency $\Omega = \omega_1 \pm \omega_2$.We here only write down the sum-frequency force contributions connected to the term $\frac{\partial}{\partial t}(\nabla\Phi^{(1)} \cdot \nabla\Phi^{(1)})$ in the free surface non-homogeneity, called the A-part by Sclavounos. After interchanging the order of the integration,being legal due to the exponential decay of the integrands, typical terms for the force may be written

$$X_{i_{AD}} = \frac{i\rho\omega_1\omega_2 A_1 A_2(\omega_1+\omega_2)^2 r^2}{2g} \int_{-\infty}^{\infty} du \frac{\nu_1+\nu_2-u}{(|u|-N_\delta^+)(|u-\nu_1|-\nu_2^\epsilon)} H_i(u) S(u) \quad (2)$$

where $H_i(u)$ is the classical Kochin function given by

$$H_i(u) = \int_0^{2\pi} d\alpha (n_i - \psi_i \frac{\partial}{\partial n}) e^{|u|z - iux} \quad (3)$$

The normal vector $\vec{n} = (n_1, n_2)$ and $n_3\vec{j} = (x, z) \times \vec{n}$ where $\vec{j}$ is the normal vector pointing out of the plane. $S(u)$ may be called the "source" Kochin function and is given by

$$S(u) = \int_0^{2\pi} d\alpha_2 \sigma_2(\alpha_2) e^{|u-\nu_1|\zeta_2 + i(u-\nu_1)\xi_2} \quad (4)$$

Here σ_i is the linear source strength,A_i is the amplitude of the incoming waves and ω_i is the corresponding frequency, i=1,2.

$$\begin{aligned} X_{i_{AR}} &= -\frac{\rho\omega_1\omega_2(\omega_1+\omega_2)^2 A_1 A_2 r^2}{2g} \\ & \int_0^{\infty} du \frac{1}{u-N_\delta^+} \Big(\int_0^{\infty} du_1 \frac{(\nu_1\nu_2 + u_1^2 + uu_1) F1(u,u_1)}{(u_1-\nu_1^\epsilon)(u+u_1-\nu_2^\epsilon)} \\ &+ \int_u^{\infty} du_1 \frac{(\nu_1\nu_2 + u_1^2 - uu_1) F2(u,u_1)}{(u_1-\nu_1^\epsilon)(u_1-u-\nu_2^\epsilon)} \\ &+ \int_0^{u} du_1 \frac{(\nu_1\nu_2 + u_1^2 - uu_1) F3(u,u_1)}{(u_1-\nu_1^\epsilon)(u-u_1-\nu_2^\epsilon)} \Big) \end{aligned} \quad (5)$$

Here F1,F2 and F3 consists of a combination of Kochin functions. For example

$$F1(u, u_1) = H_{1i}(u) S_{11}(u_1) S_{21}(u, u_1) + H_{2i}(u) S_{12}(u_1) S_{22}(u, u_1) \quad (6)$$

with the following Kochin functions

$$H_{1i}(u) = \int_0^{2\pi} d\alpha(n_i - \psi_i(iun_1 + un_2))e^{uz+iux} \quad (7)$$

$$H_{2i}(u) = \int_0^{2\pi} d\alpha(n_i - \psi_i(un_2 - iun_1))e^{uz-iux} \quad (8)$$

$$S_{11}(u_1) = \int_0^{2\pi} d\alpha_1\sigma_1(\alpha_1)e^{u_1\zeta_1+iu_1\xi_1} \quad (9)$$

$$S_{12}(u_1) = \int_0^{2\pi} d\alpha_1\sigma_1(\alpha_1)e^{u_1\zeta_1-iu_1\xi_1} \quad (10)$$

$$S_{21}(u,u_1) = \int_0^{2\pi} d\alpha_2\sigma_2(\alpha_2)e^{\zeta_2(u+u_1)-i\xi_2(u+u_1)} \quad (11)$$

$$S_{22}(u,u_1) = \int_0^{2\pi} d\alpha_2\sigma_2(\alpha_2)e^{\zeta_2(u+u_1)+i\xi_2(u+u_1)} \quad (12)$$

Similar expressions exists for F2 and F3. The force expressions are integrated by 2-point Gaussian quadrature. The inner integrals in (5) are evaluated using a rectangular path in the complex u_1-plane.

The second order forces have been calculated for a circular cylinder for both one and two incoming waves.For one incoming wave the results have been compared to Vada(1987).The results agree very well with Vada's results. In the method used by Vada it is required a high accuracy of the first order solution which is not the case in the present method.

A cruical point is if the present method is more efficient than the standard method used by Vada.That has not been the case in these computations. However,it seems reasonable that the computation of the double integrals used here may be considerably effectivised.

2 The second order reflection coeficcient

It is a well known fact that the first order reflection coeficcient R_1 is zero for a circular cylinder.By using this second order theory it can be proved that the same is true for second order reflection coeficcient R_2, defined as the (normalized) amplitude of the outgoing second order free wave with wavenumber 4ν.

Letting $x \rightarrow -\infty$ the particular potential ϕ_p simplifies obtaining only contributions from the residues.

For the circular contour the "source" Kochin functions S_{12}, S_{21} and the other functions with a similar structure can be shown to be zero.This is obtained by writing the linear source strength as a Fourier series on the contour

$$\sigma(\alpha) = \sum_{m=1}^{\infty}(a_m cos(m\alpha) + b_m sin(m\alpha)) \quad (13)$$

where a_m and b_m are complex quantities.

We can then for instance write the Kochin function S_{12} as

$$S_{12}(u_1) = \pi e^{-hu_1} \sum_{n=1}^{\infty} (a_n + ib_n) \frac{u_1^n}{n!} \tag{14}$$

where the parametrization $\xi = -sin(\alpha), \zeta = -h + cos(\alpha)$ has been used.
It can now be shown that the Fourier coeficcients fulfills the relation $a_m + ib_m = 0$,by examening the integral equation for the source strength.But this is exactly what is needed to prove that these "source" Kochin functions are zero and thereby it is easily seen that ϕ_p vanish as $x \rightarrow -\infty$.
To prove that ϕ_h is zero as $x \rightarrow -\infty$,we write ϕ_h as a source distribution over the submerged body. Writing the source strength in the form (13),we again find that $a_m + ib_m = 0$.
When examening the expression for ϕ_h as $x \rightarrow -\infty$

$$\phi_h = 2\pi i e^{4\nu z + i4\nu x} \int_0^{2\pi} d\alpha \sigma_h e^{4\nu\zeta - i4\nu\xi} \tag{15}$$

with the Fourier expression substituted for σ_h,it is easily shown that ϕ_h vanish.
Hence we have proved that R_2=0 for a circular cylinder.

References

[1] SCLAVOUNOS, P.D Radiation and diffraction of second order surface waves by floating bodies. *J. Fluid Mech. Vol 196,pp 65-91* 1988.

[2] VADA,T A numerical solution of the second order wave diffraction problem for a submerged cylinder of arbitary shape. *J. Fluid Mech. Vol 174,pp 23-37* 1987.

DISCUSSION

Palm: Based on your experience, please comment on the advantages and disadvantages of using the method of Sclavounos.

Friis: The main advantage is that it gives explicit expressions for the contribution to the second-order forces due to the second-order potential which do not involve an integration of the free-surface non-homogeneity. Moreover, in these expressions, the first-order solution is not required to high accuracy. However, it is difficult to achieve a fast numerical computation of these expressions (involving integrals), at least without further extensive analysis.

McIver: A comment: the result that there is no reflection at second order from a submerged circular cylinder in deep water has been proved, using a different method, by M. McIver & P. McIver [*J. Fluid Mech.*, to appear].

Friis: This is very interesting. I look forward to seeing your proof and comparing it with mine.