Mean forces on floating bodies in waves and current

John Grue and Enok Palm

Department of Mechanics, University of Oslo

Knowledge of mean wave forces when a current is present is of importance in the ship and offshore industry. One example is the slow drift damping mechanism, which is the usually positive change in the mean wave drift force on a body due to a small increase in the current speed. Another example is the added resistance on high speed vessels or sailboats due to incoming waves. Many earlier works have treated this theme. However, we feel that a more thorough analysis is needed.

Let us consider a body oscillating in incoming monochromatic waves with amplitude a on a current. Let $\mathbf{v}$ denote the velocity of the fluid, composed by the current speed $\mathbf{U}$ and the perturbation $\mathbf{v}'$, which is due to the presence of the body, i.e. $\mathbf{v} = \mathbf{U} + \mathbf{v}'$. The perturbation $\mathbf{v}'$ contains all first and higher order components of the velocity field. We will show that the mean force can be determined by first order (in wave amplitude) quantities only, in the case when there is no circulation in the fluid.

The momentum equation

Let us introduce horizontal coordinates x and y in the mean free surface, and let z be vertical upwards. Let p denote the pressure in the fluid and ρ the fluid density. Furthermore, let S_∞ denote a vertical non-moving control surface far away from the body, extending from the sea bottom to the instantaneous free surface, and let $\mathbf{n}$ be unit normal of this surface, pointing out of the fluid. Then the momentum equation gives the mean force $\mathbf{F}$ on the body by

$$\mathbf{F} = -\overline{\int_{S_\infty} (p\mathbf{n} + \rho \mathbf{v}\mathbf{v} \cdot \mathbf{n}) dS} \tag{1}$$

There is no integral over the free surface since we can assume without loss of generality that the pressure there is zero. Let us assume that there is a velocity potential ϕ such that $\mathbf{v}' = \nabla\phi$. Then the pressure in the fluid is given by

$$p = -\rho(\phi_t + \mathbf{U} \cdot \mathbf{v}' + \frac{1}{2}\mathbf{v}'^2 + gz) + C(t) \tag{2}$$

We are considering a situation where the fluid motion inside S_∞ has no transients, i.e. the motion is composed of a steady part and and oscillatory part. Thus we have that $\overline{\phi_t} = 0$ and $C(t) = O(a^2)$. Then the momentum equation becomes

$$\mathbf{F} = -\overline{\int_{S_\infty}(-\rho(\phi_t + \frac{1}{2}\mathbf{v}'^2 + gz)\mathbf{n} + \rho\mathbf{v}'\mathbf{v}'\cdot\mathbf{n})dS} - \overline{C(t)\int_{S_\infty}\mathbf{n}dS}$$
$$-\rho\mathbf{U}\overline{\int_{S_\infty}\mathbf{v}\cdot\mathbf{n}dS} - \overline{\rho\int_{S_\infty}(\mathbf{v}'\mathbf{U}\cdot\mathbf{n} - \mathbf{v}'\cdot\mathbf{U}\mathbf{n})dS} \tag{3}$$

The first integral becomes

$$\rho\int_{C_\infty} -\frac{1}{2g}(\overline{(\phi_t)^2} - \overline{(\mathbf{U}\cdot\mathbf{v}')^2})\mathbf{n}dl + \int_{S_\infty}(\frac{1}{2}\overline{(\mathbf{v}')^2}\mathbf{n} - \overline{\mathbf{v}'\mathbf{v}'\cdot\mathbf{n}})dS \tag{4}$$

where C_∞ denotes the waterline of S_∞.

The second integral in (3) is of third order in wave amplitude and can therefore be neglected. Now, since conservation of mass requires that the integral

$$\overline{\int_{S_\infty}\mathbf{v}\cdot\mathbf{n}dS} \tag{5}$$

is equal to zero, the third integral must vanish.

Let the x-axis be oriented along the current direction. Thus, $\mathbf{U} = U\mathbf{i}$. Let $\mathbf{v}' = u'\mathbf{i} + v'\mathbf{j} + w'\mathbf{k}$ and $\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j}$ (on S_∞). Then the fourth term becomes

$$-\overline{\rho\int_{S_\infty}(\mathbf{v}'\mathbf{U}\cdot\mathbf{n} - \mathbf{v}'\cdot\mathbf{U}\mathbf{n})dS} = \rho U\mathbf{j}\overline{\int_{S_\infty}(v'n_x - u'n_y)dS} = \rho U\mathbf{j}\overline{\int_{S_\infty}\mathbf{v}'\cdot d\mathbf{l}dz} \tag{6}$$

where $d\mathbf{l}$ is horizontal and tangential to S_∞. Let us first integrate up to the mean free surface. Thus

$$\int_{S_\infty}\mathbf{v}'\cdot d\mathbf{l}dz = \int_{-h}^{0}dz\int_{C_\infty(z)}\mathbf{v}'\cdot d\mathbf{l} + \int_{C_\infty}\zeta\mathbf{v}'\cdot d\mathbf{l} \tag{7}$$

where $C_\infty(z)$ denotes a horizontal closed curve on S_∞ at height z and ζ denotes the free surface elevation. This integral is thus a horizontal lift force orthogonal to the current direction. This component may be composed by both a steady part and an oscillatory part. This component vanishes if $\mathbf{v}'$ is symmetric about the current direction.

Assuming that there is no circulation in the fluid $\int_{C_\infty}\mathbf{v}'\cdot d\mathbf{l}$ vanishes. In this case the integral reduces to

$$\rho U\mathbf{j}\int_{C_\infty}\overline{\zeta\mathbf{v}'}\cdot d\mathbf{l} \tag{8}$$

Thus, when there is no circulation in the fluid, the mean forces may be determined by first order quantities only. If there is a circulation, we need, however, to know $\mathbf{v}'$ to second order to determine the second order force.

Energy flux

The energy flux is given by

$$W = \overline{\int_{S_\infty}(p + \frac{1}{2}\rho\mathbf{v}^2 + \rho gz)\mathbf{v}\cdot\mathbf{n}dS} \tag{9}$$

Inserting for the pressure we obtain

$$W = -\rho\overline{\int_{S_\infty}\phi_t\mathbf{v}\cdot\mathbf{n}dS} - \frac{1}{2}\rho\mathbf{U}^2\overline{\int_{S_\infty}\mathbf{v}\cdot\mathbf{n}dS} + \overline{C(t)\int_{S_\infty}\mathbf{v}\cdot\mathbf{n}dS} \tag{10}$$

The second integral vanishes due to conservation of mass, and the third integral is of third order in wave amplitude and can be neglected. Hence,

$$W = -\rho\int_{S_\infty}\overline{\phi_t\mathbf{v}'}\cdot\mathbf{n}dS - \rho\int_{C_\infty}\overline{\phi_t\zeta}\mathbf{U}\cdot\mathbf{n}dl \tag{11}$$

Thus, the energy flux at S_∞ is determied by first order quantities. These conclusions for **F** and W hold for arbitrary water depth and current speed.

Application for small U

Let h denote a constant water depth, and let us consider **F** for small current speed **U**. Far away from the body the motion will then be composed by an incoming wave field given by the potential

$$\phi_0 = \Re e^{i\sigma t}\frac{Aig}{\omega}\frac{\cosh K(z+h)}{\cosh Kh}e^{-iK(x\cos\beta+y\sin\beta)} \tag{12}$$

where $\omega^2 = gK\tanh Kh$, $\omega = \sigma - UK\cos\beta$, and β denotes angle of incidence relative to the positive x-axis. A radiated wave field is given by

$$\phi_R = \Re e^{i\sigma t}R^{-\frac{1}{2}}H(\alpha)\frac{\cosh\kappa_1(z+h)}{\cosh\kappa_1 h}e^{-i\kappa_1 R(1+O(\tau^2))} \tag{13}$$

where $\tau = \frac{U\sigma}{g}$, $x = R\cos\alpha, y = R\sin\alpha$, $H(\alpha)$ is amplitude distribution, and

$$\kappa_1 = \kappa_0(1 - \frac{2\tau\cos\alpha}{\tanh Kh + \frac{Kh}{\cosh^2 Kh}} + O(\tau^2)) \tag{14}$$

κ_0 is given by $\sigma^2 = g\kappa_0\tanh\kappa_0$.

Let C_∞ be circular with radius R. Then we have in this case that $\mathbf{v}_R'\cdot d\mathbf{l} = \frac{\partial\phi_R}{\partial\alpha}d\alpha = O(\tau)$. Furthermore, ϕ_R interfer with ϕ_0 only for $K\cos(\beta-\alpha) = \kappa_1(1+O(\tau^2))$. Thus, to leading order (6) is $O(U^2)$. Hence, **F** is given by (4) for small current speed **U**.

DISCUSSION

Hu: Following the discussions on the mean moment (yaw), I have shown in my Ph.D. thesis [University of London, 1989] that the mean moment is given by a similar formula as the mean forces, for small forward speed. The key is to use the generalized normal component n_6.

Grue: To us, it is not yet clear that the expression for the moment in yaw will be similar to that of the force. It rather looks like the moment will include a steady coupling between the forward speed and a steady, second-order velocity.

Hu: For finite forward speed it is important to approximate the force integral on the basis of the steady wave profile, which may not be small, certainly not at $z = 0$. In particular, the total surface elevation ζ is not a small quantity. Thus, your derivation may not be appropriate for finite forward speed in general; see, e.g. your equation (7). However, I agree with the final expressions for small forward speed.

Grue: In many 3-D examples, equation (7) will be appropriate. However, for very large set-downs, (7) has to be modified. In that context, the lift force component may be important.