OSCILLATING IMMERSED PLATES AND
HYPERSINGULAR INTEGRAL EQUATIONS, II

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1. Introduction

Consider the two-dimensional problem of a thin plate immersed in water; the plate is oscillated in calm water, or it is held fixed while a given surface wave is incident upon it. At the Third Workshop, we described methods for reducing these problems to hypersingular integral equations over the plate (Martin, 1988):

$$\oint_{\Gamma} [\phi(q)] \frac{\partial^2}{\partial n_p \partial n_q} G(p,q)\, ds_q = V(p), \tag{1}$$

for p on Γ. Here, Γ is the plate, p and q are points on Γ, $V(p)$ is a known function, $\partial/\partial n_q$ denotes normal differentiation at q, $[\phi]$ is the discontinuity in the velocity potential ϕ across the plate, and G is the usual fundamental solution. Equation (1) is a hypersingular integral equation for $[\phi]$; the integral must be interpreted as a finite-part integral:

$$\oint_{-1}^{1} \frac{f(t)}{(x-t)^2}\, dt = \lim_{\epsilon \to 0} \left\{ \int_{-1}^{x-\epsilon} \frac{f(t)}{(x-t)^2}\, dt + \int_{x+\epsilon}^{1} \frac{f(t)}{(x-t)^2}\, dt - \frac{2f(x)}{\epsilon} \right\}$$

where $-1 < x < 1$ and f is required to have a Hölder-continuous derivative.

For a derivation of (1) (actually for the Helmholtz equation rather than Laplace's equation), see Martin & Rizzo (1989). This paper also includes a discussion of numerical methods for treating (1). Higson (1988) has used (1) to compute the reflection and transmission coefficients for scattering by a submerged flat plate, inclined at various angles.

In the derivation of (1), it is assumed that

$$[\phi(q)] \to 0 \quad \text{as} \quad s \to 0, \tag{2}$$

where s is the distance from q to the edge of the plate; this assumption is appropriate for submerged plates. We expect more than (2), namely

$$[\phi(q)] \sim A s^{1/2} \quad \text{as} \quad s \to 0, \tag{3}$$

where A is a constant. Moreover, this information should be used in any efficient numerical treatment of (1).

In fact, the edge behaviour (3) can be extracted from the governing integral equation itself. To do this, our principal tool is the *Mellin transform*, defined by

$$\widetilde{f}(z) = \int_0^\infty f(t) t^{z-1}\, dt.$$

Mellin transforms are often used to find asymptotic expansions of integrals. For example, Bleistein & Handelsman (1975, Chapter 4) show how to obtain asymptotic approximations of

$$I(\lambda) = \int_0^\infty h(\lambda t) f(t)\, dt \tag{4}$$

for small or large values of $|\lambda|$, where h and f are known functions. However, we can view (4) as an integral equation for f: given I and h, we can use Mellin transforms to find the asymptotic behaviour of $f(t)$ near $t = 0$. We propose to use this method on (1).

More interesting problems obtain if the plate intersects the free surface or if there is a flow-induced wake; in these problems, $[\phi]$ is only required to be bounded at one edge of the plate (rather than (2)). There are also analogous problems when the plate occupies a two-dimensional surface in a three-dimensional ocean. Some of these problems are currently under investigation; some preliminary results are sketched below.

2. Submerged flat plate

Consider a flat plate, so that Γ is parametrized as

$$\Gamma: \quad x = t \sin\alpha, \quad y = d + t\cos\alpha, \quad 0 \le t \le 1,$$

where the plate is inclined at an angle α to the vertical ($|\alpha| \le \frac{1}{2}\pi$) and d is the distance between the top edge and the mean free surface ($y = 0$) of deep water (in $y > 0$). The integral equation (1) can be written as

$$\frac{1}{2\pi} ⨍_0^1 f(t) L(t,\tau)\, dt = v(\tau), \quad 0 < \tau < 1, \tag{5}$$

where

$$L(t,\tau) = \frac{1}{(t-\tau)^2} + \frac{Y^2 - X^2}{(X^2+Y^2)^2} + \frac{2KY}{X^2+Y^2} + 2K^2 ⨍_0^\infty e^{-kY} \cos kX \frac{dk}{k-K},$$

$X = (t-\tau)\sin\alpha$, $Y = (t+\tau)\cos\alpha + 2d$, $K = \omega^2/g$, $f(t) = [\phi(q)]$ and $v(\tau) = V(p)$.

We shall consider two special cases of (5), namely, a plate in unbounded fluid and a vertical surface-piercing plate. In each case, we determine the behaviour of $f(t)$ as $t \to 0$ from $\widetilde{f}(z)$.

3. Mellin transforms

We extend $f(t)$ by zero for $t > 1$, whence $\widetilde{f}(z)$ exists and is analytic in a right-hand plane. In fact, we can be more precise.

THEOREM (Bleistein & Handelsman, 1975, Lemma 4.3.6)
Suppose that $f(t) = 0$ for $t > 1$ and

$$f(t) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{N(m)} A_{mn} t^{a_m} (\log t)^n \quad as \quad t \to 0+,$$

where $\mathrm{Re}(a_0) \leq \mathrm{Re}(a_1) \leq \cdots$ *and* $0 \leq N(m)$, *finite. Then* $\widetilde{f}(z)$ *is analytic in* $\mathrm{Re}(z) > -\mathrm{Re}(a_0)$ *and can be analytically continued into* $\mathrm{Re}(z) \leq -\mathrm{Re}(a_0)$, *with poles at* $z = -a_m$. *Also, the principal part of the Laurent expansion of* $\widetilde{f}(z)$ *about* $z = -a_m$ *is*

$$\sum_{n=0}^{N(m)} A_{mn} \frac{(-1)^n n!}{(z + a_m)^{n+1}}.$$

So, there is a direct connection between the poles of $\widetilde{f}(z)$ and the expansion of $f(t)$ for small t.

4. Flat plate in unbounded fluid

For a plate in unbounded fluid (formally, let $d \to \infty$), (5) simplifies to

$$\frac{1}{2\pi} \fint_0^1 f(t) \frac{dt}{(y-t)^2} = v(y), \quad 0 < y < 1. \tag{6}$$

We know that $f(0) = f(1) = 0$, whence $\widetilde{f}(z)$ is analytic in $\mathrm{Re}(z) \geq 0$ and has poles in $\mathrm{Re}(z) < 0$. $v(y)$ is given for $0 < y < 1$; assume that

$$v(y) = \sum_{n=0} v_n y^n \qquad \text{for small } y.$$

Define $v(y)$ for $y > 1$ by the left-hand side of (6), whence $v(y) \sim y^{-2}$ as $y \to \infty$. Thus, $\widetilde{v}(z)$ is analytic in $0 < \mathrm{Re}(z) < 2$ and can be analytically continued into the whole plane apart from poles; in particular, there are simple poles at $z = -n$ ($n = 0, 1, 2, \ldots$), with residue v_n. Taking the Mellin transform of (6), we obtain

$$\widetilde{f}(z) = \frac{-2}{\cos \pi z} \frac{\sin \pi z}{z} \widetilde{v}(z+1). \tag{7}$$

Note that this is not a *formula* for $\widetilde{f}(z)$, since $\widetilde{v}$ depends on $\widetilde{f}$. However, we know that the right-hand side can only have singularities at the (simple) zeros of $\cos \pi z$. Since $\widetilde{f}(z)$ is analytic for $\mathrm{Re}(z) \geq 0$, we deduce that $\widetilde{v}(n + \frac{3}{2}) = 0$ for $n = 0, 1, 2, \ldots$. Moving the inversion contour to the left, the first pole that we meet is the simple pole at $z = -\frac{1}{2}$, whence

$$f(t) \sim A t^{1/2} \quad \text{as} \quad t \to 0$$

in agreement with our expectation (3); the coefficient $A = -(4/\pi)\widetilde{v}(1/2)$. Note that we also get higher-order terms; the next terms are $t^{3/2}$, $t^{5/2}$, etc.

5. Surface-piercing vertical barrier

Suppose now that we set $d = 0$ and $\alpha = 0$ in (5). We are interested in the behaviour of $f(t)$ as $t \to 0$, i.e. of $[\phi(q)]$ as q approaches the point where the vertical plate meets the free surface. We assume that $f(t)$ is bounded as $t \to 0$, whence $\widetilde{f}(z)$ is analytic in

$\text{Re}(z) > 0$ and has poles in $\text{Re}(z) \leq 0$; also, we can only allow a *simple* pole at $z = 0$, otherwise f would be logarithmically infinite at $t = 0$. Taking the Mellin transform of the integral equation, we find that

$$z \sin^2\left(\frac{\pi}{2}z\right)\tilde{f}(z) = -KQ(z) + \tilde{v}(z+1)\sin\pi z \tag{8}$$

where

$$Q(z) = \frac{e^{i\pi z}}{K^z}\Gamma(z+1)\int_0^1 f(\eta)e^{-K\eta}\,d\eta - \sum_{n=0}^{\infty}\frac{\Gamma(z+1)(-K)^n}{\Gamma(z+n+1)}\tilde{f}(z+n+1).$$

Again, (8) is not a formula for $\tilde{f}$. This time, it relates $\tilde{f}(z)$ to its value at points to the right of z and to $\tilde{v}$. The idea now is to step towards the left, using a 'bootstrap' argument, i.e. as we step, we deduce information about $\tilde{f}(z)$ which we then use in the right-hand side of (8) so that we can can step further to the left.

We start by moving the inversion contour to the left, meeting the first pole at $z = 0$; since

$$Q(z) = A_0 + A_1 z + A_2 z^2 + O(z^3) \quad \text{as} \quad z \to 0,$$

it appears that $\tilde{f}$ has a triple pole at $z = 0$. However, this is an illusion, for it turns out that $A_0 = 0$ and $-KA_1 + \pi\tilde{v}(1) = 0$ (from the integral equation), whence $\tilde{f}(z)$ has a *simple* pole at $z = 0$, as required. Next, we deduce that Q, and hence $\tilde{f}$, has a *simple* pole at $z = -1$. Then we deduce that $\tilde{f}$ has a *triple* pole at $z = -2$. The result of these singularities in $\tilde{f}$ is that, in general, f behaves according to

$$f(t) \sim A + Bt + t^2\left(C(\log t)^2 + D\log t + E\right) \quad \text{as} \quad t \to 0, \tag{9}$$

where A, B, C, D and E are constants.

References

1. N. Bleistein & R.A. Handelsman, *Asymptotic Expansions of Integrals*, Holt, Rinehart & Winston, New York, 1975.
2. H.L. Higson, *Diffraction of Water Waves by Submerged Plates*, M.Sc. Dissertation, University of Manchester, 1988.
3. P.A. Martin, 'Oscillating immersed plates and hypersingular integral equations', *Proc. 3rd Int. Workshop on Water Waves & Floating Bodies*, Woods Hole (1988) 117–119.
4. P.A. Martin & F.J. Rizzo, 'On boundary integral equations for crack problems', *Proc. Roy. Soc.* A**421** (1989) 341–355.

DISCUSSION

Mehlum: A comment: the information that you collect about the behaviour at the ends of the plate can be used to find rapidly converging series based on Jacobi polynomials. For example, if $f(t) \sim t^{1/2}$ as $t \to 0$ and $f(t) \sim (1-t)^{1/2}$ as $t \to 1$, then one should use Chebyshev polynomials of the second kind, U_n.

Martin: I agree. This works well because of the formula

$$\frac{1}{\pi} \int_{-1}^{1} \frac{(1-t^2)^{1/2}}{(x-t)^2} U_n(t)\, dt = -(n+1)U_n(x).$$

This method has been used for one-dimensional hypersingular integral equations; for some references, see my paper with Frank Rizzo.

Sclavounos: Is it likely that, in the diffraction problem for a plate intersecting the free surface at right angles, the diffraction solution is more regular at the intersection than in the sway-radiation problem? For example, if the plate is of infinite depth the solution is a standing wave, which is analytic at the origin.

Martin: Yes. The result (9) does not preclude the possibility that some of the coefficients might vanish (since they depend on the data, $v(y)$ for $0 < y < 1$). For the diffraction problem, the total potential ϕ satisfies

$$\phi_x = 0 \quad \text{on} \quad \Gamma_R \equiv \{x = 0, y > 0\} \quad \text{and} \quad K\phi + \phi_y = 0 \quad \text{on} \quad F_R \equiv \{y = 0, x > 0\}.$$

Define $\psi = K\phi + \phi_y$. Then, $\psi = 0$ on F_R and $\psi_x = 0$ on Γ_R. We can continue ψ, by reflection, and deduce that it is single-valued around O. In particular, we can recover ϕ on Γ_R, assuming that it is bounded at O: it has an expansion in ascending integer powers of y. (This continuation argument was suggested to me by Fritz Ursell). This result can be verified by analysing Ursell's exact solution [1] for scattering by a vertical barrier. Note that the argument is local, i.e. the result does not depend on the length of the barrier. The same result obtains for the radiation problem, provided that v satisfies

$$Kv(y) + v_y(y) = 0 \quad \text{for} \quad 0 \le y \le c,$$

for some $c > 0$, otherwise the weak singularities displayed in (9) will be present.

Reference

[1] F. Ursell, 'The effect of a fixed vertical barrier on surface waves in deep water', *Proc. Camb. Phil. Soc.* **43** (1947) 374–382.