# THE SLOW-DRIFT OSCILLATION OF A FLOATING BODY

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## SUMMARY

The derivation is outlined of a time-variant ordinary differential equation governing the rectilinear slow-drift oscillation of a three-dimensional body in deep-water random surface waves. Perturbation theory is employed to decouple the rapid small amplitude body oscillations from the large amplitude slowly varying motion. The slow-drift damping force is time dependent, it appears parametrically in the equation of motion and is governed by a linear and a second-order free-surface problem. The properties of the slow-drift equation of motion are discussed and the use of consistent pairs of second-order excitation and damping forces in narrow and wide band wave spectra is emphasized.

## THE BOUNDARY-VALUE PROBLEMS

Consider a freely floating body of mass $M$ undergoing a slow-drift oscillation along the $X$-axis of an inertial coordinate system, under the action of unidirectional random waves and the linear restoring force $CX$ of a weak mooring mechanism. Assuming that the characteristic frequency of the ambient waves is large compared to $\sqrt{C/M}$, we may define the body slow-drift velocity $U(t) = \dot{X}_0(t) = O(U)$ by averaging its horizontal displacement over a time interval of several wave periods small relative to the period of the slow drift oscillation.

Denoting by $L$ the body dimension, two *independently* small parameters are defined, the characteristic wave slope $\delta$ and the Froude number $F = U/\sqrt{gL}$, where $g$ is the gravitational acceleration. Ignoring the effects of viscosity, the ideal wave flow around the body is governed by the velocity potential $\Phi(\vec{X},t)$ subject to a normal velocity condition on the body boundary and the nonlinear free-surface condition

$$\frac{d^2\Phi}{dt^2} + g\Phi_Z + 2\nabla\Phi \cdot \frac{d\nabla\Phi}{dt} + \frac{1}{2}\nabla\Phi \cdot \nabla\left(\nabla\Phi \cdot \nabla\Phi\right) = 0 \tag{1}$$

enforced on $z = \varsigma(X,Y,t)$,

$$\varsigma = -\frac{1}{g}\left(\frac{d\Phi}{dt} + \frac{1}{2}\nabla\Phi \cdot \nabla\Phi\right)_{z=\varsigma}. \tag{2}$$

Introduce the translating coordinate system $(x,y,z) = (X - X_0(t), Y, Z)$, the velocity potential $\phi(x,y,z,t) = \Phi(X,Y,Z,t)$ and the relation $d^n\Phi/dt^n = (\partial/\partial t - U\partial/\partial x)^n\phi, n = 1,2$. The substitution of these definitions in (1)-(2) allows the expression of the exact nonlinear problem with respect to the translating frame. The following perturbation expansions are assumed for the velocity potential and wave elevation,

$$\phi(\vec{x},t) = \underbrace{\phi_{01}(\vec{x},t)}_{O(F)} + \underbrace{\phi_{10}(\vec{x},t)}_{O(\delta)} + \underbrace{\phi_{11}(\vec{x},t)}_{O(\delta F)} + \underbrace{\phi_{20}(\vec{x},t)}_{O(\delta^2)} + \underbrace{\phi_{21}(\vec{x},t)}_{O(\delta^2 F)} + \cdots \tag{3}$$

$$\varsigma(x,y,t) = \underbrace{\varsigma_{10}(x,y,t)}_{O(\delta)} + \underbrace{\varsigma_{11}(x,y,t)}_{O(\delta F)} + \underbrace{\varsigma_{20}(x,y,t)}_{O(\delta^2)} + \underbrace{\varsigma_{21}(x,y,t)}_{O(\delta^2 F)} + \underbrace{\varsigma_{01}(x,y,t)}_{O(F^2)} + \cdots \quad (4)$$

Introducing (3) and (4) in the exact free-surface problem expressed with respect to the translating frame, and expanding formally about the $z = 0$ plane, it is possible to derive a sequence of boundary-value problems governing the velocity potentials $\phi_{ij}$. Only the free-surface conditions will be discussed here since they usually cause the greater difficulty in the solution of the respective boundary-value problems.

The velocity potential $\phi_{01}$ describes the disturbance double-body flow and is therefore subject to a rigid wall free-surface condition. The zero-speed potential $\phi_{10}$ satisfies the familiar linear free-surface condition, while the second-order potential $\phi_{20}$ is subject to its second-order extension which has been extensively studied [e.g. Ogilvie (1983)]. Effects due to the slow-drift velocity $U$ are present in the linear and second-order potentials $\phi_{11}$ and $\phi_{21}$ which are subject to the free-surface conditions on $z = 0$

$$\frac{\partial^2\phi_{11}}{\partial t^2} + g\frac{\partial\phi_{11}}{\partial z} = \frac{\partial\phi_{10}}{\partial t}\frac{\partial^2\phi_{01}}{\partial z^2} + 2U\frac{\partial^2\phi_{10}}{\partial x\partial t} - 2\nabla\phi_{01}\cdot\frac{\partial\nabla\phi_{10}}{\partial t}, \quad (5)$$

$$\begin{aligned}\frac{\partial^2\phi_{21}}{\partial t^2} + g\frac{\partial\phi_{21}}{\partial z} &= -\frac{1}{2}\nabla\phi_{01}\cdot\nabla(\nabla\phi_{10}\cdot\nabla\phi_{10}) - \nabla\phi_{10}\cdot\nabla\left(\nabla\phi_{01}\cdot\nabla\phi_{10} - 2U\frac{\partial\phi_{10}}{\partial x}\right)\\ &-\frac{\partial}{\partial z}\left[\varsigma_{10}\left(\frac{\partial^2\phi_{11}}{\partial t^2} + g\frac{\partial\phi_{11}}{\partial z}\right) + \varsigma_{11}\left(\frac{\partial^2\phi_{10}}{\partial t^2} + g\frac{\partial\phi_{10}}{\partial z}\right)\right]\\ -2\nabla\phi_{11}\cdot\frac{\partial}{\partial t}\nabla\phi_{10} - 2\nabla\phi_{10}\cdot\frac{\partial}{\partial t}\nabla\phi_{11} &- 2\nabla\phi_{01}\cdot\frac{\partial}{\partial t}\nabla\phi_{20} - 2\varsigma_{10}\frac{\partial}{\partial z}\left(\nabla\phi_{10}\cdot\frac{\partial}{\partial t}\nabla\phi_{10}\right) \quad (6)\end{aligned}$$

where $\varsigma_{10}$ is the linear wave elevation and $\varsigma_{11}$ is defined by

$$\varsigma_{11} = -\frac{1}{g}\left(\frac{\partial\phi_{11}}{\partial t} - U\frac{\partial\phi_{10}}{\partial x} + \nabla\phi_{10}\cdot\nabla\phi_{01}\right)_{z=0}. \quad (7)$$

The velocity potential $\phi_{11}$ represents the leading-order forward-speed correction to the linear problem for small slow-drift velocities $U$, while $\phi_{21}$ represents the corresponding effects in the second-order problem. The free-surface condition (5) may also be derived from the formulation of the forward-speed ship motion problem presented by Newman (1978), and has been the basis of most studies of the slow-drift damping. The free-surface condition (6) is to our knowledge new and significantly more difficult to treat numerically due to the presence of up to third order spatial derivatives of lower order potentials. The details of the derivation of (6) and of the body boundary conditions enforced at the mean position of the body boundary are presented in Sclavounos (1990), where the effects of a slow-drift yaw oscillation are also included.

In the derivation of (5)-(7) from (1)-(2) it was assumed that $\partial/\partial t = O(1)$ and $\dot{U} = O(F^2)$, in accordance with the assumption that the flow contains two time scales with characteristic periods differing by $O(F)$ [Triantafyllou (1982), Agnon, Choi and Mei (1986), Chen and Molin (1989)]. The order of magnitude of the slow-drift velocity $U(t)$ and its time rate of change are here defined by the properties of the mass-spring oscillator formed by the body and the mooring system. Assuming that the amplitude of the slow-drift oscillation is of $O(L)$, we obtain $U = O(L\sqrt{C/M}) = O(F)$. The time rate of change of the velocity potentials $\phi_{ij}(t)$ has not yet been restricted in any way.

## THE HYDRODYNAMIC FORCES

The total hydrodynamic force experienced by the body as it undergoes a slow-drift oscillation depends on the velocity potentials $\phi_{ij}$ and their gradients in a manner which may be inferred from the form of the hydrodynamic pressure at some point in the fluid domain. It follows from Bernoulli's equation and the expansion (3) that

$$p(\vec{x},t) = \underbrace{p_{00}(\vec{x})}_{O(1)} + \underbrace{p_{10}(\vec{x},t)}_{O(\delta)} + \underbrace{p_{11}(\vec{x},t)}_{O(\delta F)} + \underbrace{p_{20}(\vec{x},t)}_{O(\delta^2)} + \underbrace{p_{21}(\vec{x},t)}_{O(\delta^2 F)} + \underbrace{p_{01}(\vec{x},t)}_{O(F^2)} + \cdots \tag{8}$$

where, $p_{00}$ is the hydrostatic, $p_{10}$ the linear, and $p_{01}$ the pressure due to the double-body disturbance. The remaining components are defined by the relations

$$p_{11} = -\rho\left[\frac{\partial\phi_{11}}{\partial t} - U\frac{\partial\phi_{10}}{\partial x} + \nabla\phi_{10}\cdot\nabla\phi_{01}\right] \tag{9}$$

$$p_{20} = -\rho\left[\frac{\partial\phi_{20}}{\partial t} + \frac{1}{2}\nabla\phi_{10}\cdot\nabla\phi_{10}\right] \tag{10}$$

$$p_{21} = -\rho\left[\frac{\partial\phi_{21}}{\partial t} - U\frac{\partial\phi_{20}}{\partial x} + \nabla\phi_{01}\cdot\nabla\phi_{20} + \nabla\phi_{11}\cdot\nabla\phi_{10}\right]. \tag{11}$$

The utility of the indicial notation $p_{ij}$ is now clear in that it allows the easy identification of the velocity potential combinations which must appear in the definition of the pressure component of $O(\delta^i F^j)$. The pressure on the body boundary includes additional terms which arise from a Taylor series expansion around its mean position. It follows that the total hydrodynamic force in the $x$−direction may also be expanded in the form

$$H(t) = \underbrace{H_{10}(t)}_{O(\delta)} + \underbrace{H_{11}(t)}_{O(\delta F)} + \underbrace{H_{20}(t)}_{O(\delta^2)} + \underbrace{H_{21}(t)}_{O(\delta^2 F)} + \underbrace{H_{01}(t)}_{O(F^2)} + \cdots \tag{12}$$

where the dependence of the different force components on the velocity potentials $\phi_{ij}$ is similar to that in relations (9)-(11). Separating out the terms which account for the slow-drift velocity corrections, we may write

$$H(t) = F(t) - \dot{X}_0(t)B(t) \tag{13}$$

where the "excitation" force $F(t)$ is the sum of the $H_{i0}$ terms and the "damping" coefficient $B(t)$ is the sum of the $H_{i1}$ terms of $O(F)$ for unit $U = \dot{X}_0(t)$.

## THE SLOW-DRIFT EQUATION OF MOTION

Enforcing Newton's law for the body motion in the $X$−direction, and averaging out over several wave periods the hydrodynamic and inertia, we obtain

$$(M + A)\ddot{X}_0(t) + \dot{X}_0(t)B_{SLOW}(t) + CX_0(t) = D_{SLOW}(t) + F_{SLOW}(t) \tag{14}$$

where $A$ is the body added mass due to the double-body flow and $D_{SLOW}$ is the slowly varying inertia force which arises from nonlinear combinations of the fast rotational body oscillations.

In a wave spectrum described by the superposition of linear wave components, the "slow" component of the wave force is often defined as the collection of all difference-frequency force components. This definition will be adopted in the ensuing discussion, emphasizing however that it does not allow for a slow force component to arise from linear wave effects. In practice this assumption is usually satisfactory.

It follows that the slow-drift excitation and damping forces $F_{SLOW}$ and $B_{SLOW}$ in (14) include only second-order wave effects. The former may me obtained from a pressure equation similar to (10) and requires the solution of the linear and second-order velocity potentials $\phi_{10}$ and $\phi_{20}$. The latter requires the determination of $\phi_{11}$ and $\phi_{21}$, and by virtue of (11) also depends on gradients of $\phi_{20}$. Therefore, if no assumption is invoked about the ambient wave spectrum, the free-surface problem (6) must be treated. Moreover, knowledge of the second-order potential $\phi_{20}$ is necessary for the determination of the slow-drift damping.

In narrow band spectra, the Newman approximation suggests that the $\partial/\phi_{20}\partial t$ derivative in (10) operating on the slow component of the pressure is small relative to the remaining terms. The second-order excitation force $F_{SLOW}$ may then be expressed in terms of the mean drift force, while it remains time dependent over a time scale comparable to the period of the slow-drift oscillation. Similar arguments apply to the slow-drift damping $B_{SLOW}$ governed by the pressure equation (11). The solution of the free-surface problem (6) for $\phi_{21}$ may be unnecessary in narrow band spectra, yet the determination of gradients of $\phi_{20}$ appear to be important.

Concluding, it is appropriate to emphasize two properties of the problem. First, the slow-drift damping must be regarded as a time-dependent quantity over the time scale of the slow-drift oscillation both in narrow and wide-band spectra. Secondly, consistent pairs of excitation and damping forces must be employed. In a narrow-band spectrum the drift force is sufficient for the evaluation of the former force, while the solution of $\phi_{11}$ and the zero-frequency limit of $\phi_{20}$ is necessary for the evaluation of the slow-drift damping. In wide-band spectra all potentials up to and including $\phi_{21}$ must be determined.

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## DISCUSSION

**Grue**: Let me first congratulate you on your nice solution for the cylinder case! Next, I want to remark that there are at least two papers [1, 2] that discuss the effect of time-dependent damping. Their conclusions are that time-dependent damping is important, and is larger than the mean drift damping (which is the one that is usually applied).

**Sclavounos**: I am aware of those papers and their mention of the importance of the time-dependent nature of the slow-drift damping. Yet, I am under the impression that they did not attempt to solve the boundary-value problem for $\phi_{21}$ in the context mentioned in my abstract.

### References


[1] R. Zhao & O.M. Faltinsen, 'A comparative study of theoretical models for slowdrift sway motion of a marine structure', OMAE, Houston (1988).

[2] J. Grue, 'Wave drift damping and low-frequency oscillations of an elliptic cylinder in irregular waves', *Appl. Ocean Res.* **10** (1988) 10–19.