Trapped modes near bodies in channels

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ABSTRACT

Trapped modes over submerged long horizontal cylinders are well known. Thus Ursell [1] demonstrated the existence of trapped modes over a submerged horizontal circular cylinder of sufficiently small radius whilst Jones [2] produced general results for arbitrarily-shaped submerged horizontal cylinders symmetric about a vertical plane through their axis. Further results were described at the last workshop by Callan [3] and Bonnet & Joly [4].

In this work it is proved that trapped modes exist in the vicinity of a vertical circular cylinder in a wave tank at frequencies below the cut-off frequency for the tank, for cylinders of sufficiently small radius. The method is an extension of that used by Ursell [1] for submerged horizontal cylinders in which an expansion in terms of multipole potentials is used to express the condition for trapped modes in terms of the vanishing of the determinant of a certain infinite system of equations.

Numerical computations for any size cylinder suggest that there is just one trapped mode frequency with a corresponding motion which is antisymmetric with respect to the centre-line of the channel and symmetric with respect to a vertical plane through the centre of the cylinder perpendicular to the channel walls.

Results will also be presented for the trapped mode frequencies above an infinite line of identical submerged sea-mounts. Longuet-Higgins [5] showed that it was possible to construct 'leaky' modes above a single submerged circular sea-mount. Here we show that if there are an infinite line of such mounts, a genuine trapped mode exists, on linear theory, which does not radiate energy to infinity.

Formulation and solution

We seek a function $\phi(x,y)$ satisfying

$$(\nabla^2+k^2)\phi(x,y) = 0 \quad \text{in} \quad r > a,\ |y| < d,\ r = (x^2+y^2)^{\frac{1}{2}} \tag{1}$$

$$\phi_y = 0, \quad |y| = d,\ -\infty < x < \infty \tag{2}$$

$$\phi_r = 0, \quad r = a \tag{3}$$

$$\phi = 0, \quad y = 0,\ |x| \geq a \tag{4}$$

$$\phi \longrightarrow 0, \quad |x| \longrightarrow \infty,\ |y| \leq d \tag{5}$$

Thus ϕ can be regarded as a time-independent acoustic potential, the actual potential being derived from $\mathcal{R}e\ \phi \exp(i\omega t)$ where $k = \omega/c$ and c is the velocity of sound. Equivalently as in Evans and Linton [6] the equations describe a water-wave problem in which a vertical cylinder extends throughout the depth H thereby permitting a depth dependence cosh k(z+H) to be separated out from the governing Laplace's equation. In this case k is the positive real root of

$$\omega^2 = gk \tanh kH \tag{6}$$

Note from (4) that we seek solutions which are antisymmetric with respect to the central line of the channel.

We construct multipole potentials satisfying conditions (1), (2), (4) and (5) which are singular at the origin, in the form

$$\psi_{2n+1}(r,\theta) = Y_{2n+1}(kr)\sin(2n+1)\theta + \mathcal{R}e\,\frac{(-1)^n}{\pi}\int_{-\infty}^{\infty+i\pi} \frac{e^{\gamma d}}{\cosh\gamma d}\sinh\gamma y\cos(kx\cosh v)e^{-(2n+1)v}dv \tag{7}$$

where $x = r\cos\theta$, $y = r\sin\theta$, $\gamma = k\sinh v$ and the contour is taken along the negative real axis, up the imaginary axis to $i\pi$ and then along the line $i\pi + s$, $s > 0$. It can be shown that for $r > 0$

$$\psi_{2n+1}(r,\theta) = Y_{2n+1}(kr)\sin(2n+1)\theta + \sum_{m=0}^{\infty} A_{mn}J_{2m+1}(kr)\sin(2m+1)\theta \tag{8}$$

where

$$A_{mn} = -\frac{4(-1)^{m+n}}{\pi}\int_0^{\pi+2} \frac{e^{-\gamma d}\sinh(2n+1)v\sinh(2m+1)v}{\cosh\gamma d}dv - \frac{4}{\pi}\int_0^{\pi/2} \tan(\beta d)\cos(2n+1)u\cos(2m+1)u\,du \tag{9}$$

where $\beta = k\cos u$.

We seek a trapped mode solution in the form

$$\phi(r,\theta) = \sum_{n=0}^{\infty} k^{-1}a_n(Y'_{2n+1}(ka))^{-1}\psi_{2n+1}(r,\theta) \tag{10}$$

Application of the condition (3), multiplication by $\sin(2m+1)\theta$ and integration over the cylinder results in the homogeneous infinite system of equations

$$a_m + \sum_{n=0}^{\infty} B_{mn}a_n = 0, \quad m = 0,1,2,\ldots \tag{11}$$

where

$$B_{mn} = A_{mn}\frac{J'_{2m+1}(ka)}{Y'_{2n+1}(ka)} \tag{12}$$

A sufficient condition for the determinant Δ_N of the truncated system

$$a_m + \sum_{n=0}^{N} B_{mn}a_n = 0, \quad m = 0,1,\ldots N \tag{13}$$

to converge uniformly to Δ_∞ the infinite determinant of the system (11), is

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |B_{mn}| < \infty$$

and it can be shown that this is satisfied for

$$0 < ka < kd < \frac{\pi}{2}, \text{ and } \coth\chi < M(ka)^{-2} \text{ where} \tag{14}$$

$kd\cosh\chi = \frac{1}{2}\pi$, and M is independent of ka, kd.

We can also conclude that the infinite system behaves in all respects like a finite system and in particular has a non-trivial solution with $\Sigma|a_m| < \infty$ if and only if

$$\Delta_\infty = \det(\delta_{mn} + B_{mn}(ka,kd))$$

vanishes for some ka, kd, such that (14) is satisfied.

If we write $(ka)^2 = \frac{1}{2}\lambda\tanh\chi$, λ fixed, and let $ka \rightarrow 0$ so that $kd \rightarrow \frac{\pi}{2}$ or $\chi \rightarrow 0$, we find

$$B_{mn} \sim - \frac{(2m+1)\lambda(ka)^{2m+2n}}{(2n+1)(2n+1)!}$$

so that all elements of $B_{mn} \rightarrow 0$ except B_{oo}. We find that

$$\Delta_\infty(ka,\lambda) = \det(\delta_{mn} + B_{mn}(ka,\lambda)) \rightarrow 1 - \lambda$$

so that Δ_∞ vanishes if $(ka)^2 \sim \frac{1}{2}\tanh\chi$ as $ka \rightarrow 0$, $kd \rightarrow \frac{\pi}{2}$. It is easy to show that the corresponding potential exists and is non-zero so that we have constructed a trapped mode provided the cylinder is sufficiently small.

The details are given in Callan et al. [7] and the authors are grateful to Fritz Ursell for considerable assistance in some of the finer points of the analysis.

Computation of the real solutions of the infinite determinant when the cylinder is small is straightforward using a conventional library routine. The results are shown in Fig.1.

The theory is easily modified to illustrate numerically the existence of genuine trapped modes around an infinite line of submerged sea-mounts or correspondingly, a sea-mount in a channel. This provides an extension of the work of Longuet-Higgins [5] who used shallow-water theory to construct 'leaky' trapped modes near a submerged sea-mount in an infinite ocean. Here the presence of the channel walls ensure that the modes are genuinely trapped and no energy, however small, leaks away to infinity.

On shallow water theory the expansions in the region $r > a$ are the same as before, but now we have an inner region $r < a$ in which we have a potential ϕ_1 satisfying

$$(\nabla^2 + k_1^2)\phi = 0, \quad r < a$$

where $k_1^2 = \omega^2/gh$, and h is the depth of water above the sea-mount. The appropriate solution is

$$\phi_1 = \sum_{n=0}^{\infty} b_n J_{2n+1}(k_1 r)\sin(2n+1)\theta$$

and the matching conditions on $r = a$ are $\phi_1 = \phi$ and $h\phi_{1r} = H\phi_r$, resulting in a slightly more complicated infinite system of equations. Results for the trapped mode wave numbers are given in Fig.2.

There is no technical reason why the shallow water theory need be used and work is already in hand which predicts trapped modes for both a submerged sea-mount and a partly immersed truncated cylinder, using full linear theory.

References

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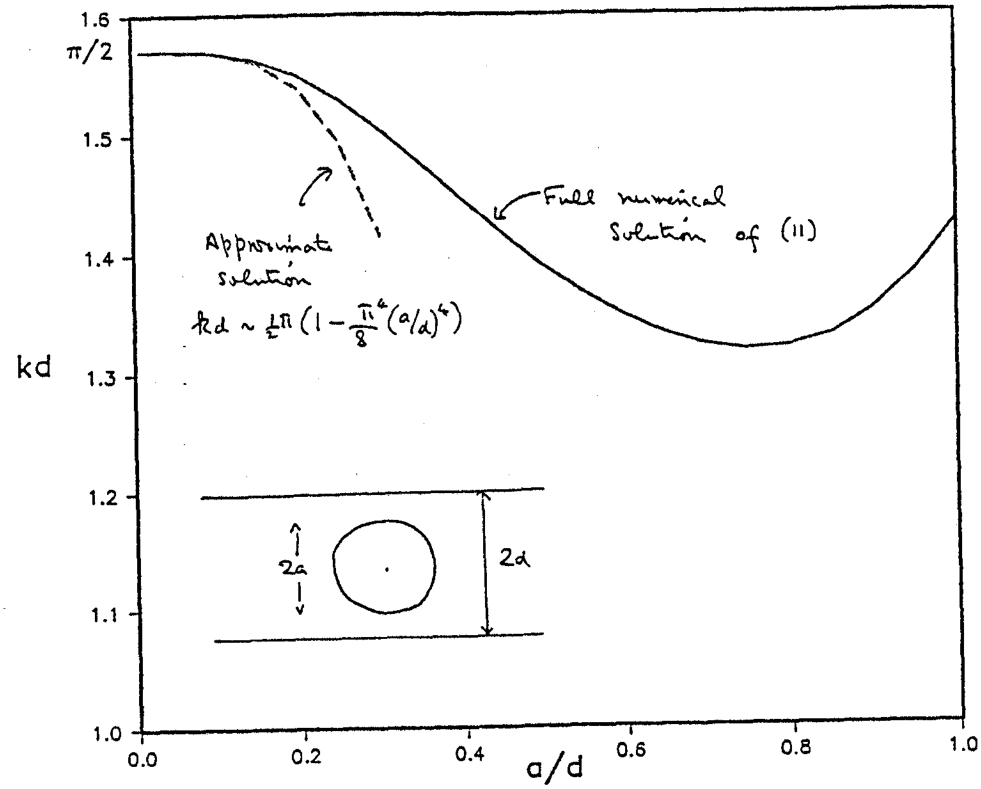
1.6
π/2
1.5
1.4
1.3
1.2
1.1
1.0
kd
0.0
0.2
0.4
0.6
0.8
1.0
a/d
Approximate solution
$kd \sim \frac{1}{2}\pi\left(1-\frac{\pi^4}{8}(a/d)^4\right)$
Full numerical Solution of (11)
2a
2d

Fig. 1

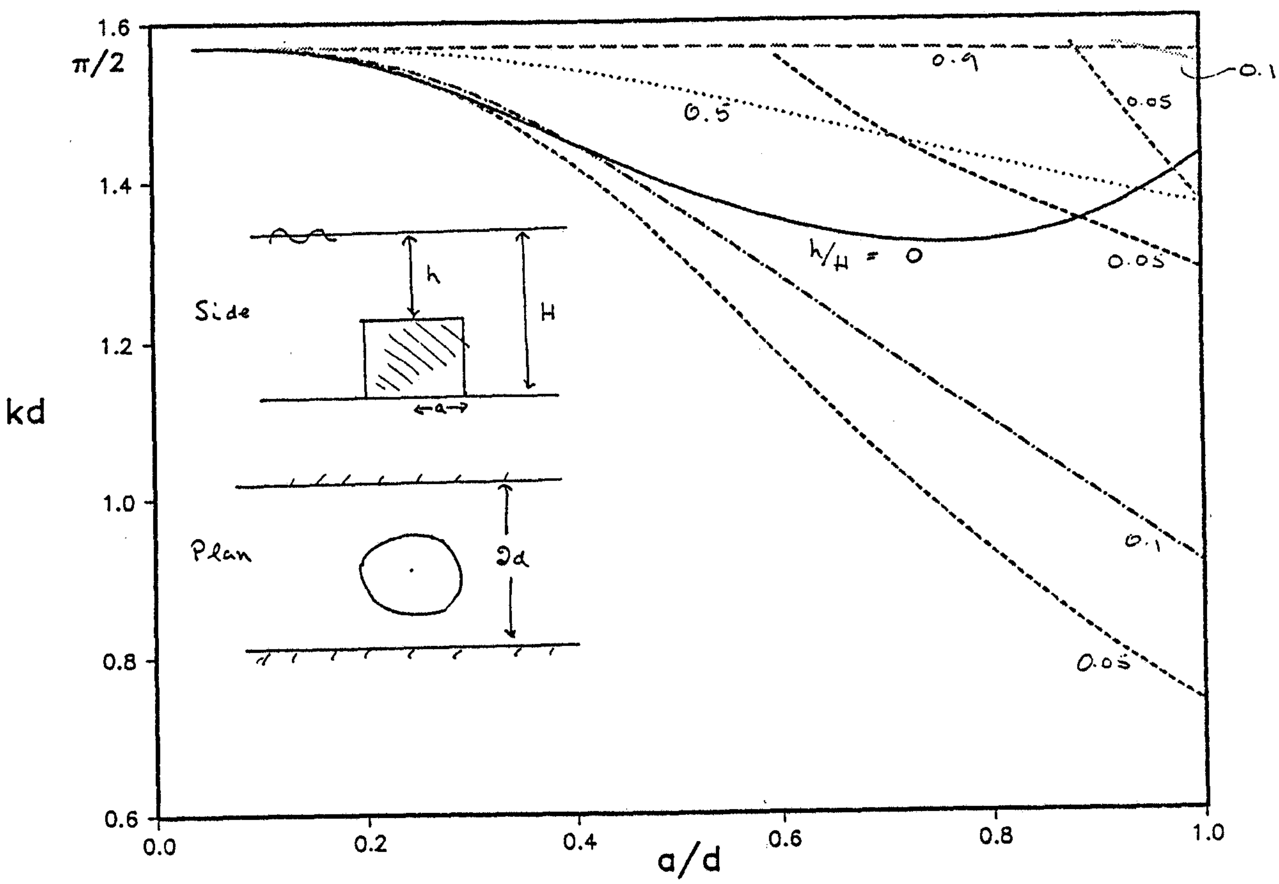
1.6
π/2
1.4
1.2
1.0
0.8
0.6
kd
0.0
0.2
0.4
0.6
0.8
1.0
a/d
0.9
0.5
0.1
0.05
0.05
h/H = 0
0.1
0.05
Side
h
H
a
Plan
2d

Fig. 2.

Newman: Is there a logical connection between a transverse periodic array of compact bodies, such as the vertical cylinder, and a transverse infinitely long cylinder? If so it might explain the presence of trapped modes in the former case.

Evans: It is perhaps not surprising that a periodic array of submerged sea-mounts should exhibit trapped modes by comparison with the Ursell long horizontal submerged cylinder, but since the method of construction is so different in each case that it is not clear what the connection is. What is more surprising is the predicted existence of trapped modes for a periodic array of truncated surface-piercing cylinders, since the comparison in this case would be to a long partly immersed cylinder for which no trapped modes exist. However, the analogy is not strong and the variable free surface geometry in the former case is clearly important for trapped modes.

Mei: Does the existence of trapped modes around vertical cylinders in a channel imply that irregular frequencies exist in the numerical solution (using integral equations) of diffraction by the same cylinder?

Evans: I believe that irregular frequencies are the frequencies corresponding to eigenfunctions of the interior problem with a Dirichelet boundary condition, and therefore have no physical reality in the exterior problem. They are products of the method used to solve the problem. The trapped modes discussed here are genuine solutions to the homogeneous exterior problem. No doubt an integral equation formulation would break down at both the trapped mode frequencies and the irregular frequencies but only the former have physical significance.

Tulin: We operate a wavemaker with a large 3D pump on it. Will trapped modes result?

Evans: Only if the wavemaker is symmetric about the centerline of the channel and makes anti-symmetric motions about that line at frequencies below the cutoff can one be confident that trapped mode frequencies may result. A lot of work needs to be done to determine precisely when to expect them.

Yeung: We have done calculations, using a matching method, for a truncated cylinder in a finite-width tank. We had no difficulties with the heave and diffraction problems. Nor, as I recall, did we have any problems computing the sway added mass and damping coefficients at frequencies below the cut off. We used a very fine frequency discretization and would probably have detected the trapped modes had they occurred. Should trapped modes be present, it would appear that they would contribute a singular behavior to the added mass, but not to the damping, since the trapped mode energy cannot be radiated down the tank. In any event, there is a possibility that the geometrical combinations were such that the trapped-mode frequency could be very close to the cut-off frequency, and hence "shadowed" by the cut-off effects. Also, to my recollection, more wild behavior than that which could be associated with trapped modes, occurs at higher frequencies, near the higher-mode 'cut-offs.' Your analysis does not appear to predict trapped modes above the first cut-off, or does it? Our work was published in J. Eng. Math., 1989, and PRADS, 1989 Symposium at Varna, for two types of problems.

Evans: No, our theory only predicts trapped modes below the first cut-off and as you say, this could well be masked by the cut-off frequency for the geometry under consideration. We are currently determining trapped mode frequencies for both truncated surface cylinders and submerged sea-mounts using the full linear theory. Preliminary investigations suggest that for typical geometries the frequencies are close to the first cut-off and could well be overlooked. Our latest ideas lead us to believe that there may well exist trapped mode frequencies in situations in which there is no natural cut-off frequency, that is they would be imbedded in the continuous spectrum. For example, using methods similar to that described by Linton in his paper at this workshop, we think that we can show the existence of trapped modes near a thin vertical plate positioned off the center-line, parallel to the channel walls, and extending throughout the depth. We hope to report on this at the next workshop.