Ursell's multipoles and the Rayleigh hypothesis

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1 Introduction

Ursell's multipoles were invented for the problem of a heaving, half-immersed circular cylinder [6]. The velocity potential can be expanded as an infinite series of these multipoles, and this series converges everywhere in the water and on the cylinder's wetted surface, S.

Suppose, now, that S is not a semicircle. We still have the set of multipoles, each one of which satisfies all conditions of the boundary-value problem, save one: can they be combined so as to satisfy the boundary condition on S? In other words, can the potential be represented *everywhere in the water* as a convergent series of multipoles, with coefficients determined by the boundary condition on S? In general, the answer is 'no'. However, for *some* geometries and for *some* forcings, the representation is valid. Here, we give a method for determining the allowable geometries and forcings. This method is an adaptation of a method due to van den Berg and Fokkema [7] for determining the limitations of the so-called *Rayleigh hypothesis* in the theory of gratings. Three examples are given (elliptic cylinder, partially-immersed circular cylinder and 'squashed' circular cylinder). We have previously given similar results for Havelock wavemakers, in which waves are generated in a semi-infinite channel of constant finite depth by a wavemaker which need not be vertical [1].

2 Formulation of the problem

Consider a horizontal cylinder, partially immersed in the free surface of deep water. Choose Cartesian coordinates (x, y) and polar coordinates (r, θ), with

$$x = r\sin\theta, \quad y = r\cos\theta,$$

so that $y = 0$ is the mean free surface, y increasing with depth. We assume that the wetted surface of the cylinder, S, is given by

$$S: \ r = \rho(\theta), \quad -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}, \quad \text{with } a \le \rho \le b,$$

for some constants a and b, and take the fluid domain as

$$D: \ r > \rho(\theta), \quad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}.$$

For simplicity, we consider symmetric time-harmonic motions generated by the forced heaving of a symmetric cylinder ($\rho(\theta) = \rho(-\theta)$). Under the usual assumptions, we seek a velocity potential in the form $\mathrm{Re}\,\{\phi(x,y)e^{-i\omega t}\}$, where

$$\nabla^2\phi = 0 \quad \text{in the water } D, \tag{1}$$

$$K\phi + \frac{\partial\phi}{\partial y} = 0 \quad \text{on the free surface, } y = 0,\ |x| > \rho(\tfrac{\pi}{2}), \tag{2}$$

$$\frac{\partial\phi}{\partial n} = U \quad \text{on the cylinder, } S \tag{3}$$

and ϕ corresponds to outgoing waves as $|x| \to \infty$. Here, $K = \omega^2/g$, $\partial/\partial n$ denotes normal differentiation into D and $U(\theta)$ is the prescribed normal velocity on S; for a heaving rigid cylinder,

$$U(\theta) = U_0\{\cos\theta + w(\theta)\sin\theta\}\,(1 + w^2)^{-1/2},$$

where U_0 is a constant and

$$w(\theta) = \rho'(\theta)/\rho(\theta).$$

3 The half-immersed circular cylinder: Ursell's multipoles

For a half-immersed circular cylinder, $\rho(\theta) = a = b$ and $w(\theta) \equiv 0$. Then, ϕ can be represented *everywhere in the water and on the cylinder* by [6]

$$\phi(r,\theta) = \sum_{n=0}^{\infty} c_n a^{2n} \Phi_n(r,\theta), \tag{4}$$

where $\{\Phi_n\}$ are *Ursell's multipole potentials*:

$$\Phi_0 = \oint_0^\infty e^{-ky} \cos kx \, \frac{dk}{k-K}, \qquad \Phi_m = \frac{\cos 2m\theta}{r^{2m}} + \frac{K}{2m-1} \frac{\cos(2m-1)\theta}{r^{2m-1}}$$

for $m = 1, 2, \ldots$ and the contour is indented below the pole so as to satisfy the radiation condition. The coefficients c_n can be determined so as to satisfy (3). This can be done by differentiating (4) term by term, and then imposing (4) at discrete points on S ('point-matching') or using a Galerkin scheme.

4 The Rayleigh hypothesis

Suppose, now, that S is not a semicircle. Can we still write ϕ as (4), where the series is uniformly convergent for all points $(r,\theta) \in D \cup W$? If so, we can try to determine c_n as before.

We call the assumption that (4) is a valid representation for ϕ in $D \cup W$ the *Rayleigh hypothesis*, as Rayleigh [4, §272a], [5], made a similar assumption in his work on acoustic scattering by a grating (an infinite, periodic corrugated surface). The Rayleigh hypothesis has generated a large literature; for a review, see [2]. It is known that the Rayleigh hypothesis is valid for some, but not all, geometries. Conditions for its validity have been devised by several authors. Here, we show that the method of van den Berg & Fokkema [7] can be adapted to the present problem.

It is clear that we can expand ϕ as (4) for $r \geq b$: one merely imagines that there is a certain (unknown) variation of $\partial\phi/\partial r$ over the semicircle $r = b$. Indeed, this observation has been used in various 'localized finite-element methods' [3].

5 The method of van den Berg and Fokkema

We now determine sufficient conditions for the uniform convergence of (4) in the region $r \geq a$ (this region contains $D \cup S$). By the 'root test', this will be so if

$$\limsup_{n\to\infty} |c_n a^{2n} \Phi_n(a,\theta)|^{1/n} < 1. \tag{5}$$

Substituting for Φ_n, this reduces to

$$\limsup_{n\to\infty} |c_n|^{1/n} < 1. \tag{6}$$

If this holds, we can differentiate (4) term by term and apply the boundary condition (3) to give

$$\sum_{n=0}^{\infty} c_n \Psi_n(\theta) = f(\theta), \qquad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}, \tag{7}$$

where $f(\theta) = -U_0 a\{\cos\theta + w(\theta)\sin\theta\}$,

$$\Psi_m(y) = 2m\psi_{2m}(\theta)\,(a/\rho)^{2m+1} + Ka\psi_{2m-1}(\theta)(a/\rho)^{2m}, \quad m \geq 1,$$

$$\psi_n(\theta) = \cos n\theta - w(\theta)\sin n\theta,$$

and Ψ_0 is defined similarly.

Next, we determine the behaviour of c_n for large n, so that we can test (6). We do this by extending (7) to *complex* values of θ. It is sufficient to extend into the strip

$$\Theta = \{-\tfrac{\pi}{2} \leq \mathrm{Re}(\theta) \leq \tfrac{\pi}{2},\ \mathrm{Im}(\theta) \leq 0\}.$$

In this strip,

$$\lim_{n\to\infty} |\Psi_n(\theta)|^{1/n} = |\zeta(\theta)|, \tag{8}$$

where

$$\zeta(\theta) = \left(\frac{a}{\rho(\theta)}\right)^2 e^{2i\theta}. \tag{9}$$

So, instead of (7), it is natural to consider the *power series*

$$\sum_{n=0}^{\infty} c_n \zeta^n = F(\zeta), \tag{10}$$

say, in the complex ζ-plane. The radius of convergence of this series is R, where

$$R^{-1} = \limsup_{n\to\infty} |c_n|^{1/n}. \tag{11}$$

So, by (6), the Rayleigh hypothesis will be valid if $R > 1$.

The formula (9) defines a mapping from Θ into the ζ-plane. This mapping is conformal except where $\zeta' = 0$ or $\zeta' = \infty$, i.e. where

$$i - w(\theta) = 0 \tag{12}$$

or where $\rho(\theta) = 0$ or at singularities of $\rho(\theta)$. The line $\{-\frac{\pi}{2} \le \mathrm{Re}(\theta) \le \frac{\pi}{2},\ \mathrm{Im}(\theta) = 0\}$ (these values of θ correspond to S) is mapped into a closed curve C, symmetric about $\mathrm{Im}(\zeta) = 0$. On C, $|\zeta| \le 1$, whence C is strictly contained inside C_R, the circle of convergence of (10), if (6) holds.

We can find R by noting that C_R passes through the singularity of the power series (10) that is closest to $\zeta = 0$, at $\zeta(\theta_0)$, say. Thus, $\theta = \theta_0$ is either a zero of $\rho(\theta)$, or a singularity of $\rho(\theta)$ or a zero of $\zeta'(\theta)$; the latter are given by (12). Then, $R = |\zeta(\theta_0)|$ and so (11) gives

$$|\zeta(\theta_0)| > 1. \tag{13}$$

The Rayleigh hypothesis is valid if this condition is satisfied. This condition is essentially the same as that found be van den Berg & Fokkema [7] for acoustic scattering by a cylinder.

6 Three examples

Example 1

For an elliptic cylinder, given by

$$\rho(\theta) = ab(a^2 \cos^2\theta + b^2 \sin^2\theta)^{-1/2},$$

with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $b > a$ (so that a is the length of the semi-*minor* axis), the closest singularity is given by (12); explicitly, we have (cf. [7])

$$\theta_0 = -i\chi, \quad \chi \text{ real and positive, where}$$

$$e^{2\chi} = \frac{b^2 + a^2}{b^2 - a^2}.$$

Then, (13) reduces to

$$a^2/(b^2 - a^2) > 1$$

i.e. $a > \sqrt{b^2 - a^2}$, half the distance between the two foci of the ellipse. Thus, the Rayleigh hypothesis is valid if the inscribed circle to S contains these foci.

Example 2

Consider a partially immersed circular cylinder, of radius c, with its centre a distance f *below* the free surface ($f < c$); the Rayleigh hypothesis is definitely false if $f < 0$. We have

$$\rho(\theta) = f\cos\theta + \sqrt{c^2 - f^2\sin^2\theta},$$

where $a = \rho(\frac{\pi}{2}) = \sqrt{c^2 - f^2}$. We find that

$$\theta_0 = -i\chi \quad \text{where} \quad \chi = +\infty \tag{14}$$

whence $|\zeta(\theta_0)| = a^2/f^2$. There is also a corner on C at $\zeta = -1$, due to the non-normal free-surface intersections at $(\pm a, 0)$. Thus, provided

$$a > f \quad \text{i.e.} \quad f < c/\sqrt{2},$$

the series (4) will converge everywhere on S, except for the two points $(\pm a, 0)$.

Example 3

Consider the cylinder with cross-section given by

$$\rho(\theta) = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta},$$

with $a \leq b$; fixing b and reducing a corresponds to 'squashing' a half-immersed circular cylinder at the waterline. We find that θ_0 is given by (14), whence (13) reduces to

$$4a^2/(b^2 - a^2) > 1, \quad \text{i.e. } a > b/\sqrt{3};$$

this condition is sufficient for the Rayleigh hypothesis to be valid.

References

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[4] Lord Rayleigh (J.W. Strutt), *The theory of sound*, 2nd Ed., Vol.2, Macmillan & Co., London (1896).

[5] Lord Rayleigh, On the dynamical theory of gratings, *Proc. Roy. Soc. A* **79** (1907) 399–416.

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Ursell: There is another way of looking at this interesting problem. Consider *e.g.* Paul Martin's second example with the boundary condition $\partial\phi/\partial n = 0$ on the circle. Then the potential can be continued into the circle, the new boundary is the inverse of the line $y = 0$ in the circular boundary (and is thus again a circular arc, passing through $(\pm\sqrt{c^2 - f^2}, 0)$ and $(0, f)$). Actually the potential can be continued into a still wider region. If the new region of definition includes all the points exterior to the semicircle with center (0, 0) and radius $\sqrt{c^2 - f^2}$ then clearly Rayleigh's hypothesis is applicable. This argument still applies when $\psi = -Ux$ on the original boundary. A similar argument can be used for the ellipse in the first example, the method of continuation involves elliptic coordinates.

Martin: The idea is to continue the exterior potential across S into the cylinder. This continuation will have singularities. The Rayleigh hypothesis is valid if these singularities all lie inside the inscribed circle, $\rho = a$. As you correctly note, for the two examples described (partially-immersed circle and half-immersed ellipse), special methods (inversion and elliptic coordinates, respectively) can be used to show when the singularities are inside $\rho = a$. The method described in the paper can be used for arbitrary S.

Tulin: (i) I suppose that the body shape S must be analytic from the start. (ii) Wouldn't restrictions on the use of multipole expansions be alleviated if they were spread in the vertical plane inside the body instead of restricted to the origin (this is the normal procedure in aerodynamics, and the applicability problem has been somewhat studied there).

Martin: (i) Yes, except that S can have some corners on $\rho = a$, as in Example 2. (ii) Here, you want to change the problem! Presumably, there will be some limitations on the choice of S for vertical line distributions of sources, but I do not know what they are.