# MATCHING PROBLEMS INVOLVING FLOW FROM DUCTS

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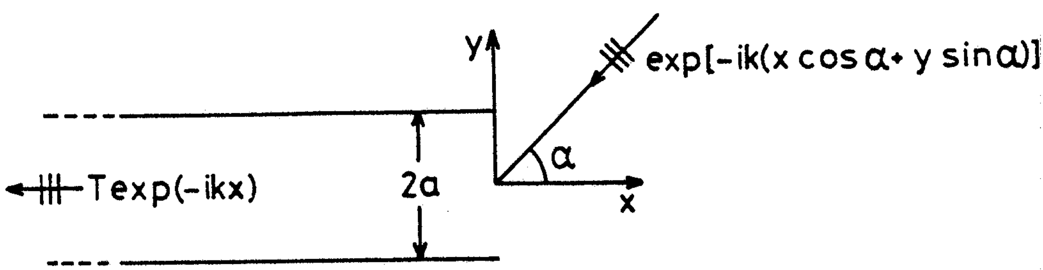
**Introduction**

The method of matched asymptotic expansions has found wide application to water-wave problems involving flow through small apertures (see Tuck, 1975). Simple approximate solutions are easily found and they often turn out to be surprisingly accurate when compared with exact solutions. A similar procedure to that introduced by Tuck has been applied to problems with parallel-walled ducts. However, in at least one case the solution is known to be incorrect. Here, the reasons for this are examined and a modified procedure is suggested.

**Scattering by a pair of semi-infinite breakwaters**

Liu (1975) used matched asymptotic expansions to calculate the transmission coefficient $T$ when a wave of wavenumber $k$ is scattered by two thin vertical breakwaters. The full geometry is illustrated below in plan view.




To determine $T$ it is sufficient to consider that part of the solution that is symmetric in $y$. The symmetric part of the incident wave potential is

$$\phi_I = \exp(-ik \cos \alpha) \cos (ky \sin \alpha) \tag{1}$$

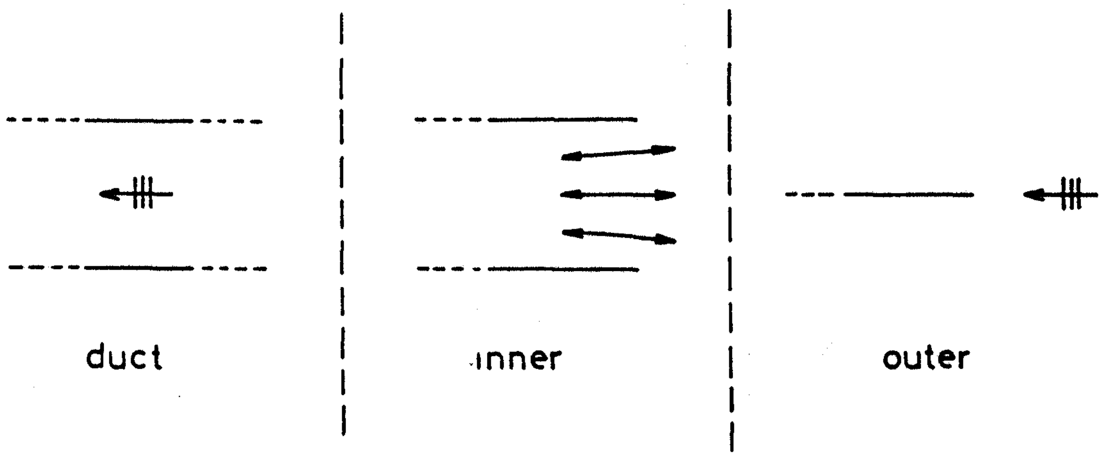
and the symmetric part of the scattered plus incident potential satisfies

$$(\nabla^2 + k^2)\phi = 0; \text{ in the fluid domain,} \tag{2}$$

$$\frac{\partial \phi}{\partial y} = 0; \; y = a, \; x < 0, \tag{3}$$

$$\frac{\partial \phi}{\partial y} = 0; \; y = 0, \tag{4}$$

together with appropriate radiation conditions. A harmonic time dependence and the depth dependence have been removed and, because of the symmetry, attention is restricted to $y \geq 0$. A solution is sought under the assumption that the wavelength is much longer than the duct spacing, that is $\varepsilon \equiv ka \ll 1$. The fluid domain is divided into three regions, as illustrated below.




In the outer region the boundary condition on the breakwaters may be transferred onto $y = 0$ by Taylor expansion. Put $a = \varepsilon a_1$ then

$$\frac{\partial\phi}{\partial y} + \varepsilon a_1 \frac{\partial^2\phi}{\partial y^2} + O(\varepsilon^2) = 0;\ y = 0,\ x < 0 \tag{5}$$

or, to a first approximation,

$$\frac{\partial\phi}{\partial y} = 0;\ y = 0,\ x < 0. \tag{6}$$

Liu's argument was essentially as follows. In the outer region the first approximation to the flow is the incident wave and a source of unknown strength at the origin to model the flow from the duct, thus

$$\phi = \phi_I + Q\ H_0(kr) = 1 + Q\,(1 + \frac{2i}{\pi}(\ln\frac{kr}{2} + \gamma)) + O((kr)^2 \ln kr),\ kr \to 0 \tag{7}$$

where $r^2 = x^2 + y^2$ and $H_0$ is the Hankel function of the first kind and order 0. In the inner region the flow scales on the length $a$ and, under the assumption $ka \ll 1$, is governed by Laplace's equation to a first approximation. The potential flow from a duct is readily determined by conformal mapping and it is found that

$$\phi \simeq A + B\ \ln\frac{\pi r}{a};\ \frac{r}{a} \to \infty\ \text{(outer limit)} \tag{8}$$

$$\phi \simeq A + B\,(\frac{\pi x}{a} - 1);\ \frac{x}{a} \to -\infty\ \text{(duct limit)} \tag{9}$$

where $A$, $B$ are unknown constants to be found from matching. In the duct region the solution is simply

$$\phi = T\ e^{-ikx} = T\ (1 - ikx) + O((kr)^2),\ kx \to 0. \tag{10}$$

Matching like terms in (7) and (8) and in (9) and (10) gives four equations for $Q$, $A$, $B$, and $T$. Solving gives, in particular,

$$Q = -\frac{ka}{2}T,\ T = (1 + \frac{ika}{\pi}(\frac{\pi}{2i} + \ln\frac{ka}{2\pi} + \gamma - 1))^{-1}. \tag{11}$$

Rawlins (1981) has pointed out that the solution to this problem is known exactly and that this expression for $T$ is incorrect. In particular, to this order of approximation, $T$ *does* depend on the angle of incidence $\alpha$, in contrast to the result in (11).

The error arises because $Q = O(ka)$ so that the source term in (7) is an order of magnitude smaller than the incident wave term. Consequently, for consistency, terms of $O(kr)$ must be retained in the inner expansion of the incident wave. Further, the outer solution has an expansion of the form

$$\phi = \phi_0 + \varepsilon\phi_1 + \cdots \tag{12}$$

where, from (5),

$$\frac{\partial\phi_0}{\partial y} = 0, \; \frac{\partial\phi_1}{\partial y} = -a_1\frac{\partial^2\phi_0}{\partial y^2}; \; y = 0, \; x < 0 \tag{13}$$

so that (7) also requires a particular solution $\phi_P$ at order $\varepsilon = ka$ satisfying the second part of (13). The correct form of the outer solution is

$$\phi = \phi_I + \varepsilon\phi_P + Q\,H_0(kr). \tag{14}$$

A particular solution $\phi_P$ is given by Crighton and Leppington (1973) and its inner expansion has a constant leading term $(\alpha \sin\alpha)/\pi i$. The correct inner expansion of the outer solution is

$$\phi = 1 - ikx\cos\alpha - \frac{ika}{\pi}\alpha\sin\alpha + Q\left(1 + \frac{2i}{\pi}\left(\ln\frac{kr}{2} + \gamma\right)\right) + O((kr)^2 \ln kr). \tag{15}$$

To match with this a further homogeneous term is required in the inner solution so that (8) and (9) are modified to

$$\phi = -ikx\cos\alpha + A + B\ln\frac{\pi r}{a}; \; \frac{r}{a} \to \infty \tag{16}$$

$$\phi = -ikx\cos\alpha + A + B\left(\frac{\pi x}{a} - 1\right); \; \frac{x}{a} \to -\infty. \tag{17}$$

The form of the duct solution in (10) is correct at all orders so remains unchanged in this modified procedure. Matching now gives

$$T = 1 - ka\ln ka\,\frac{2i}{\pi}\sin^2\frac{\alpha}{2} + ka\left[\sin^2\frac{\alpha}{2}\left(\frac{2i}{\pi}(1 - \gamma + \ln 2\pi) - 1\right) - \frac{i}{\pi}\alpha\sin\alpha\right] + O((ka)^2), \tag{18}$$

in agreement with the exact solution.

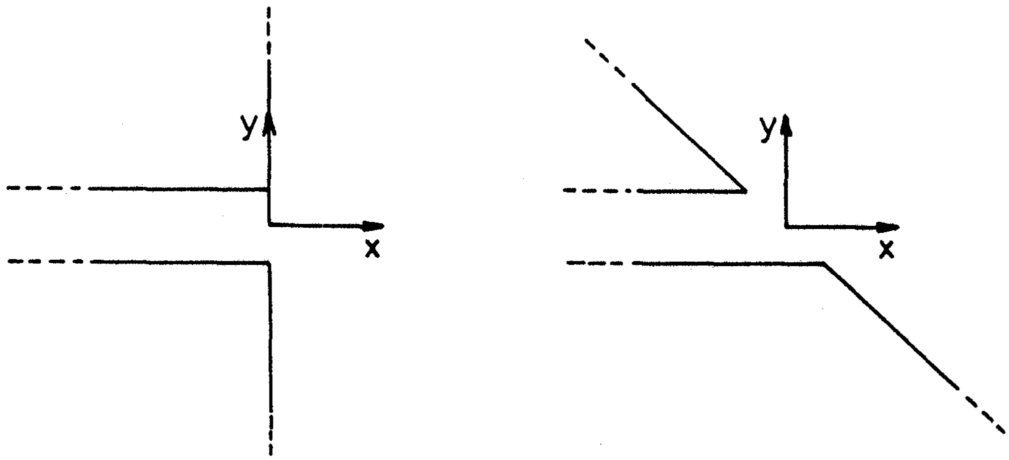
## Discussion

With a solution obtained using a simple application of the method of matched asymptotic expansions it is advisable to check that any result is consistent with the initial assumptions. In Liu's work it was assumed that the source strength $Q$ was $O(1)$, that is of a magnitude comparable with the incident wave field in (7). In fact, as can be seen from (11), $Q$ turns out to be $O(ka)$ and this may be traced back to the matching of uniform flow terms within the duct, as in (9) and (10). In the kind of problems treated by Tuck, the flow is source-like (logarithmic) on both sides of the aperture and the matching can only result in the small parameter appearing within the arguments of logarithms. To obtain the correct result in Liu's problem it is necessary to retain all $O(ka)$ terms in the outer solution and its inner expansion. Thus the solution to a non-homogeneous problem ($\phi_P$ above) and an extra term in the inner expansion of the incident wave are required for consistency.

Another duct problem that has been treated by matched asymptotic expansions is that of scattering by a perpendicular indentation in a plane wall (see Mei, 1983; illustrated below) and again the source strength is $O(ka)$. Here the leading order term in the outer solution is the the incident wave together with its reflection from the wall at $x = 0$, that is

$$\phi_I = \cos(kx\cos\alpha)\cos(ky\sin\alpha) = 1 + O((kr)^2). \tag{19}$$

For this problem there is no term at $O(kr)$ in the inner expansion of $\phi_I$. Also, there is no particular solution corresponding to $\phi_P$ in (14) so that, in this case, no modification to the procedure is needed. If the duct is not perpendicular to the wall then it is no longer sufficient to look at the symmetric part of the potential and terms of $O(kr)$ *will* appear in the required inner expansion of $\phi_I$ and a modified procedure is needed.




Newman (1974) has used matched asymptotic expansions to calculate the reflection and transmission properties of two closely spaced vertical plates extending down a finite distance from the free surface. Evans (1978) used Newman's solution and an extension to a downward facing tubular duct as a model for the oscillating column wave-energy device. In both of these works, attention was focussed on the behaviour near resonance so that in the outer solution the source modelling the flow from the duct correctly appears at the same order as the incident wave. Away from resonance the source term will be at a lower order, as in the problems discussed above, and a more careful analysis is needed.

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**Tuck:** How good is the final formula numerically relative to the exact solution?

**McIver & Rawlins:** The presented expression agrees with the expression of the exact solution. However, the form of the full exact solution (found by the Wiener-Hopf technique) is quite complex and I have not computed it.

**Evans:** There are at least four people in the room who have solved duct-type problems by matching. Which of us got it wrong?

**McIver & Rawlins:** At least two of those four analyzed duct problems involving resonant behavior as mentioned in the abstract. Near resonance the source strength is of the same order as the incident wave and, in two dimensions, the correct solution is found by Tuck's procedure developed for flow through small holes. The solution will be valid for non-dimensional frequencies within $O(ka)$ of resonance. Outside this narrow range the procedure given in the abstract is needed. In three-dimensional duct problems there are further difficulties in applying a straightforward extension of Tuck's method. These problems are still under investigation. Tuck's procedure will always work if the simple matching involves only constants and logarithms of a radial coordinate. Otherwise great care is needed.