An Explicit Form of the Impermeability Condition and its Applications in Hydrodynamics and Hydroelasticity

by

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1.0 Introduction

The impermeability condition imposed on a moving boundary of a fluid domain is usually expressed in one of the two equivalent forms: one equals to zero the substantial time derivative of the equation defining the locus of the surface in space, the other equals the component of fluid velocity along the instantaneous normal to the surface, to the normal speed of the surface, [1]. The two forms can be defined as implicit, since the displacement of the surface in space, which defines the motion of the surface, does not appear explicitly in the formulae.

The implicit formulations of the impermeability condition lead to difficulties in their applications. The difficulties are exemplified by the first derivation of the correct linear impermeability condition for a body advancing with a forward speed and oscillating about its mean configuration, in 1962, [2]. Difficulties with the application of the implicit form of the impermeability condition are particularly significant in the development of perturbation schemes for solutions to non-linear hydrodynamic problems, where the question of consistency must be considered. Also, in connection with the perturbation solutions, the independence of results from the choice of reference system is sometimes questioned. In the corresponding hydroelastic problem for a ship advancing with a forward speed, a heuristic approach was applied, [3], which unfortunately produced the incorrect result.

Below, a general explicit form of the impermeability condition is presented and used to derive a general non-linear impermeability condition applicable to the motion of a submerged elastic body. From the latter condition a linearized impermeability condition for an elastic body advancing with a constant velocity is obtained. The linearized condition is different from the condition presented in [3], but is equivalent to the condition derived in [2].

Another general explicit form of the impermeability condition can also be derived and shown to be equivalent to the one used here, [4]. The derivations of both explicit forms demonstrate that applications of the impermeability condition imply the existence of a one-to-one time dependent mapping between a reference configuration and instantaneous configurations of the impermeable boundary. It is shown below that the usual perturbation schemes used in solutions to non-linear boundary value problems formulated for floating bodies, do not allow a construction of such a mapping.

2.0 An Explicit Form of the Impermeability Condition

An explicit impermeability condition can be written, [4], as:

$$\left(\bar{v} - \frac{\partial}{\partial t}\bar{\eta}\right) \cdot \left(\bar{\bar{I}} + \frac{\partial}{\partial \bar{X}} \bullet \bar{\eta}\right)^{-1} \cdot \bar{N} = 0 \quad , \quad \text{on } S \qquad (1)$$

where $\bar{v}$ is the field of fluid velocity, $\bar{\eta}$ is the field of the wetted surface displacement from its reference configuration. $\bar{N}$ is the normal vector to the wetted surface, and $\bar{X}$ denotes the radius vector, with both vectors taken in the reference configuration of the surface, S_0. $\bar{\bar{J}}$ signifies the unit tensor. The condition is applied on the instantaneous wetted surface.

The condition can be applied on S_0, if the fluid velocity field $\bar{v}$ is developed in the Taylor's series:

$$\bar{v} = \exp\left(\bar{\eta}\cdot\frac{\partial}{\partial \bar{X}}\right)\bar{v}\Big|_{S_0} \tag{2}$$

In addition, for the use in a perturbation scheme, the inverse tensor in (1) may be represented by the Neumann's series:

$$\left(\bar{\bar{J}} + \frac{\partial}{\partial \bar{X}}\otimes\bar{\eta}\right)^{-1} = \bar{\bar{J}} - \frac{\partial}{\partial \bar{X}}\otimes\bar{\eta} + \frac{\partial}{\partial \bar{X}}\otimes\bar{\eta}\cdot\frac{\partial}{\partial \bar{X}}\otimes\bar{\eta} + \cdots \tag{3}$$

Another explicit form of the impermeability condition, which is equivalent to (1) but does not involve an inverse tensor, can also be derived, [4].

3.0 Impermeability Conditions for A Moving Elastic Body and The Kinematic Free Surface Condition

For an elastic body in motion, a body displacement from a reference configuration is given by:

$$\bar{\eta}(\bar{X},t) = \bar{\alpha}(t) + [\bar{\bar{R}}(t) - \bar{\bar{J}}]\cdot[\bar{X} - \bar{X}(0)] + \bar{\bar{R}}(t)\cdot\bar{Q}(\bar{X},t) \tag{4}$$

where $\bar{\alpha}$ is the parallel translation field, and $\bar{\bar{R}}$ is the tensor of rotation. $\bar{X}(0)$ and $\bar{Q}$ denote respectively the radius vector of the centre of rotation and the field of elastic displacement, both taken in the reference configuration. A simple manipulation gives:

$$\left(\bar{\bar{J}} + \frac{\partial}{\partial \bar{X}}\otimes\bar{\eta}\right)^{-1} = \bar{\bar{R}}\cdot\left(\bar{\bar{J}} + \frac{\partial}{\partial \bar{X}}\otimes\bar{Q}\right)^{-1}$$

and therefore, a general non-linear form of the impermeability condition for a submerged elastic body is obtained:

$$\left(\bar{v} - \frac{\partial}{\partial t}\bar{\eta}\right)\cdot\bar{\bar{R}}\cdot\left(\bar{\bar{J}} + \frac{\partial}{\partial \bar{X}}\otimes\bar{Q}\right)^{-1}\cdot\bar{N} = 0 \quad , \text{ on } S \tag{5}$$

Formula (2) can be applied to express the condition on .

For an elastic body, advancing with a velocity $\bar{U}$, condition (5) is linearized by taking:

$$\bar{\bar{R}} = \bar{\bar{J}} + \bar{\theta}\wedge\bar{\bar{J}} \qquad \text{and} \qquad \bar{\eta} = \bar{\alpha} + \bar{\theta}\wedge[\bar{X} - \bar{X}(0)] + \bar{Q}$$

where $\bar{\eta}$ is the displacement relative to a reference configuration moving with the velocity $\bar{U}$, and $\bar{\theta}$ is the vector of angular displacement. Only the first two terms of the Neumann series (3) need to be considered and (5) is reduced to:

$$\left(\bar{v} - \bar{U} - \frac{\partial}{\partial t}\bar{\eta}\right)\cdot\left(\bar{\bar{J}} + \bar{\theta}\wedge\bar{\bar{J}}\right)\cdot\left(\bar{\bar{J}} - \frac{\partial}{\partial \bar{X}}\otimes\bar{Q}\right)\cdot\bar{N} \doteq 0 \quad , \text{ on } S .$$

In addition, the following definition is introduced:

$$\bar{v} = \bar{v}' + \bar{v}_u \qquad \text{with} \quad (\bar{v}_u - \bar{U}) \cdot \bar{N} = 0 \text{ , on } S_0 .$$

With the use of (2), and after simple manipulations, the linearized impermeability condition is obtained:

$$\left[\bar{v}' - \frac{\partial}{\partial t}\bar{\eta} + \bar{\eta} \cdot \frac{\partial}{\partial \bar{X}} (\bar{v}_u - \bar{U}) - (\bar{v}_u - \bar{U}) \cdot \left(-\bar{\bar{J}} \wedge \bar{\theta} + \frac{\partial}{\partial \bar{X}} \bullet \bar{Q} \right) \right] \cdot \bar{N} = 0 \text{, on } S_0 \tag{6}$$

which is different from the condition given in [3](eq. 40). It is possible to rewrite condition (6) as:

$$\left[\bar{v}' - \frac{\partial}{\partial t}\bar{\eta} + \bar{\eta} \cdot \frac{\partial}{\partial \bar{X}} (\bar{v}_u - \bar{U}) - (\bar{v}_u - \bar{U}) \cdot \frac{\partial}{\partial \bar{X}} \bar{\eta} \right] \cdot \bar{N} = 0 \text{ , on } S_0 \tag{7}$$

and this form is equivalent to the result presented in [2] (eq. 3).

Condition (1) can also be applied if displacement field $\bar{\eta}$ is defined only on S_0. Let u_1, u_2, u_3 denote curvilinear coordinates, with the u_1 and u_2 lines on S_0, and u_3 line directed along $\bar{N}$. One has:

$$\frac{\partial}{\partial \bar{X}} \bullet \bar{\eta} = \frac{\partial u_i}{\partial \bar{X}} \bullet \frac{\partial}{\partial u_i} \bar{\eta}$$

Considering elevation $\bar{\eta}$ of the free surface $X_3 = 0$,

$$\bar{\eta} = \eta(u_1, u_2, t)\, \bar{e}_3 \qquad \text{with } u_1 = X_1 \text{ , } u_2 = X_2 \text{ and } \bar{N} = \bar{e}_3$$

it is found from (1) and (3), that:

$$v_3 = \frac{\partial}{\partial t}\eta + \bar{v} \cdot \frac{\partial}{\partial \bar{X}} \bar{\eta} \tag{8}$$

which is the well known form of the free surface kinematic condition.

4.0 The Application of The Impermeability Condition in Perturbation Formulations of Boundary Value Problems for Floating Bodies

The derivation of the explicit impermeability condition (1), and of its counterpart which is not presented here, show that applications of the impermeability condition require an explicit or implicit assumption that a one-to-one time dependent mapping exists between a reference configuration and instantaneous configurations of the impermeable boundary. In generally used perturbation formulations of non-linear boundary value problems of hydrodynamics of floating bodies, the wetted surface of a body is mapped according to its rigid body displacements, whereas the free surface of water is mapped using the single valued wave elevation mapping. This approach is illustrated in Fig 1. It is seen how each of the points a and b in an instantaneous configuration of the boundary is mapped by those mappings to two different points a′ and a″, and b′ and b″, respectively, of which points a′ and b″ do not belong to the reference boundary S_0. It is clear that the two mappings used in conjunction do not provide a one-to-one mapping between S_0 and S. Therefore it is shown that in the perturbation formulations based on those mappings the impermeability condition may not be applied correctly.

Most Frequent Formulations:

$a \leftrightarrow a'$ - rigid body mapping on the body wetted surface
$b \leftrightarrow b'$
$a \leftrightarrow a''$ - single valued elevation mapping on the free surface
$b \leftrightarrow b''$

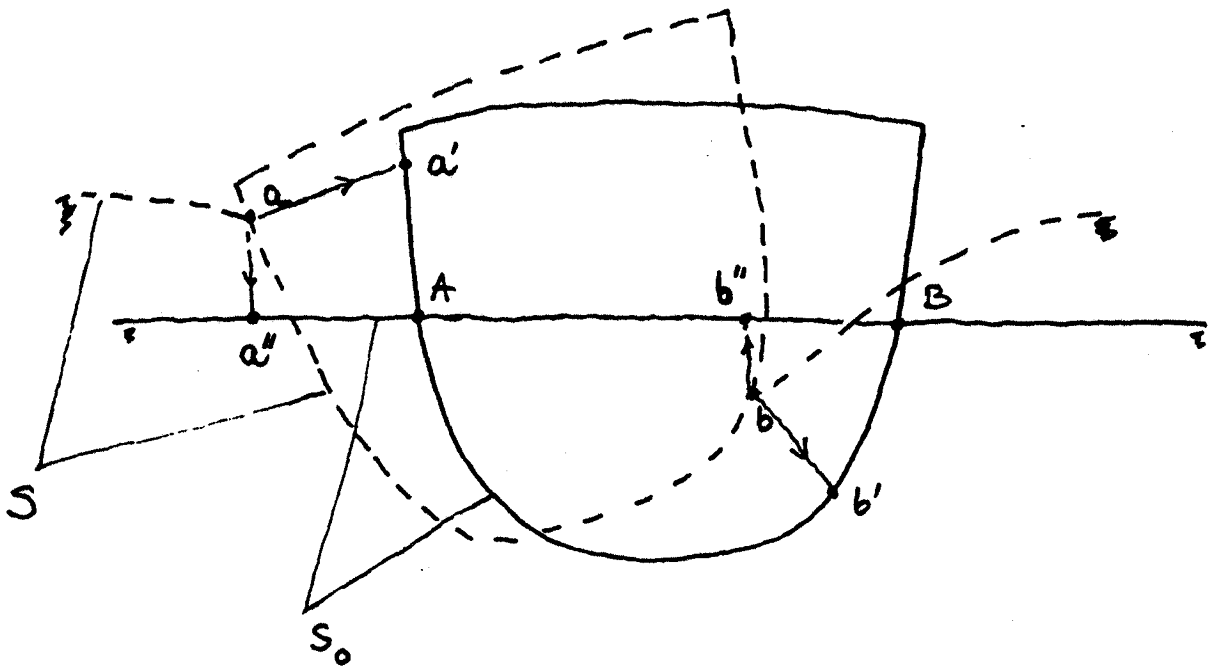

FIG. 1

5.0 References

[1] J. Serrin, "Mathematical Principles of Classical Fluid Mechanics", Encyclopedia of Physics, Vol. VIII/1, ed. S. Flugge, 1959.

[2] R. Timman and J.N. Newman, "The Coupled Dynamic Coefficients of a Symmetric Ship", Journal of Ship Research, Vol. 5, March, 1962, pp. 1-7.

[3] W.G. Price and Wu Yousheng, "Hydroelasticity of Marine Structures", Theoretical and Applied Mechanics, Elsevier Science Publishers, IUTAM 1985, pp. 311-337.

[4] J.S. Pawlowski, "A Non-Linear Theory of Ship Motion in Waves", (in preparation).

Wehausen: Is it essential to designate part of the displacement as elastic?

Pawlowski: The displacement $\overline{Q}$ in (5) and (6) may represent an arbitrary deformation of the body. However in this presentation $\overline{Q}$ was considered as an elastic deformation in view of the most probable applications of the discussed conditions.

Martin: In your abstract, you state that reference [3] contains an error. Can you be more explicit? Perhaps you could write down their incorrect condition and your correct condition, and comment on how they went wrong?!

Pawlowski: In reference [3] a formula equivalent to equation (6) with $\overline{Q} = 0$ (*i.e.*, for rigid body motion) was rederived following reference [2]. Reasoning by analogy, the rigid body angular displacement $\overline{\theta} = \overline{\theta}_r$ was augmented to $\overline{\theta} = \overline{\theta}_r + \overline{\theta}_e$, with $\overline{\theta}_e$ representing the additional, local, rotation of the body, due to the elastic displacement. The result is equivalent to replacing $\frac{\partial}{\partial X} \otimes \overline{Q}$ with $\frac{1}{2}[\frac{\partial}{\partial X} \otimes \overline{Q} - (\frac{\partial}{\partial X} \otimes \overline{Q})^T$, in equation (6). In other words, the reasoning by analogy led to the omission of the strain (*i.e.*, symmetric) component of the displacement gradient $\frac{\partial}{\partial X} \otimes \overline{\eta}$.