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WAVELESS SOLUTIONS OF WAVE EQUATIONS

by E. O. Tuck
University of Adelaide, Australia

This paper concerns a general class of equations of the form

$$\mathcal{M}y + y = F(x) \tag{1}$$

where $F(x)$ is a given function, $y = y(x)$ is to be determined, and $\mathcal{M}$ is an inertia-like operator. The prototype is $\mathcal{M} = \omega^{-2}d^2/dx^2$, where ω is a real constant, in which case (1) becomes the second-order ordinary differential equation

$$\omega^{-2}\frac{d^2y}{dx^2} + y = F(x) \tag{2}$$

describing classical forced simple harmonic motion. Almost all solutions of (2) contain periodic waves, the complementary function being $y = A\cos\omega x + B\sin\omega x$ for some arbitrary constants A, B, and we assume that a similar property holds for general inertia operators $\mathcal{M}$ in (1).

The domain of interest is semi-infinite, i.e. $0 \leq x < +\infty$, and the forcing functions $F(x)$ increase monotonically from 0 to 1 over that range, satisfying $F(0) = 0, F(\infty) = 1$. We are particularly interested in solutions y of (1) that satisfy the boundary conditions $y(0) = 0, y(\infty) = 1$. Hence, in the absence of inertia, i.e. $\mathcal{M} \equiv 0$, the solution is simply $y = F(x)$. This solution is of course not wave-like. As soon as inertia is included, waves are possible, indeed perhaps inevitable, and there are applications such as ship-generated waves (Tuck, 1990) where one might like to reduce or eliminate such waves. Are there solutions of (1) without waves, even with non-zero inertia?

For most $\mathcal{M}$, one can at best solve (1) in closed form only for very special simple choices of $F(x)$. However, the prototype equation (2) allows an explicit solution for arbitrary $F(x)$, namely

$$y = 1 + \omega\int_x^\infty [1 - F(t)]\sin\omega(x-t)dt. \tag{3}$$

This is a particular solution satisfying $y(\infty) = 1$. The general solution is obtained by adding the above wave-like complementary function, but we now rule that out, i.e. take $A = B = 0$. In order to satisfy the remaining boundary condition $y(0) = 0$, we must have

$$\omega\int_0^\infty [1 - F(t)]\sin\omega t\, dt = 1. \tag{4}$$

That is, the problem of determining waveless solutions of (2) is an eigenvalue problem. In general, i.e. for a general choice of the frequency ω, we must expect that there is no such solution. But, if we are lucky, we might find one or more special values of ω such that (4) holds, in which case waveless solutions exist at those frequencies.

For example, if $F(x) \equiv 1$, (4) formally states $0 = 1$, and there is clearly no legitimate value of ω. This is in any case obvious; $y \equiv 1$ is the unique waveless solution of (2) with $F \equiv 1$, and $y(0) = 0$ cannot be satisfied. We may wonder if (4) can ever be satisfied, but the example $F(x) = 1 - 1/x$ for which (4) is satisfied by the unique choice $\omega = 2/\pi$ quickly shows otherwise. Other less-singular examples are easy to construct, and it is possible to choose $F(x)$ so that there are any number of eigenvalues ω. Even for this simple model (2) there are many mysteries about these waveless solutions. What is it about the input function $F(x)$ that determines whether there are no eigenvalues, one eigenvalue, a finite number of eigenvalues, or infinitely many?

An alternative to (2) is $\mathcal{M} = F(x)\omega^{-2}d^2/dx^2$, corresponding (upon dividing by $F(x)$) to the DE

$$\omega^{-2}\frac{d^2y}{dx^2} + \frac{y}{F(x)} = 1. \tag{5}$$

In this case, $F(x)$ does not have a physical interpretation as an imposed external force, but rather as the reciprocal of a non-uniform restoring-force or spring coefficient. Unfortunately, for general $F(x)$, we cannot solve (5) in closed form, and hence cannot give an explicit eigenvalue equation like (4). But, so long as $F(\infty) = 1$, the equation (5) still supports waves in the limit as $x \to \infty$, and we are again interested in solutions where these waves are absent.

When $F(0) = 0$, (5) is formally singular at $x = 0$. Let us be specific about the rate at which $F(x) \to 0$ as $x \to 0$, letting $F(x) = O(x^n)$ for some $n > 0$. The singularity is "regular" if $n < 2$ and irregular if $n \geq 2$. We are interested mainly in the irregular case. Now there is a particular integral y of (5) that tends to zero at the same rate as $F(x)$, plus a complementary function, whose limiting behaviour as $x \to 0$ is of the form

$$y = x^{n/4}\left[C\cos(kx^{1-n/2}) + D\sin(kx^{1-n/2})\right] \tag{6}$$

for some constants C, D, k. When $n > 2$, this complementary function oscillates with ever-increasing frequency as $x \to 0$, while tending to zero at the rate $x^{n/4}$, which is much slower than the rate x^n at which the $F(x)$-like particular solution tends to zero. Hence the complementary function (6) is the dominant contributor to y as $x \to 0$, unless it happens that $C = D = 0$.

That is, when $F(x)$ tends to zero as $x \to 0$, all solutions of (5) already satisfy $y(0) = 0$, irrespective of the value of ω. However, almost all such solutions oscillate wildly near $x = 0$. These local oscillations may be called evanescent waves, and generally are undesirable, even perhaps unacceptable, in applications; for example, their derivative does not tend to zero as $x \to 0$. On the other hand, entirely eliminating the evanescent waves, by setting $C = D = 0$ in (6), is twice as restrictive as the single boundary condition $y(0) = 0$, and hence we cannot expect for a prescribed $F(x)$ to find solutions for any value of ω at all. Two extra disposable parameters are needed if one is to (in effect) satisfy four boundary conditions for a second-order ordinary differential equation, by demanding $A = B = 0$ as well as $C = D = 0$.

It is not obvious at first that such fully waveless solutions of (5) will ever exist. However, a simple way to generate them is by an inverse procedure, re-writing (5) as $F(x) = y_0(x)/(1 - y_0''(x)/\omega_0^2)$ and specifying the solution function $y = y_0(x)$ for some special value of $\omega = \omega_0$. Solutions $y = y(x)$ at other values of ω can then be computed by numerical solution of (5). Such computations have been carried out to high accuracy by solving the initial-value problem starting from a small but non-zero value of x with (in effect) $C = D = 0$, for some simple specifications of $y_0(x)$ and hence of $F(x)$ via the above inverse formula. The results show a single isolated zero in the outgoing wave amplitude $\sqrt{A^2 + B^2}$ as a function of ω, at $\omega = \omega_0$.

Let us now turn to integral rather than differential equations. Suppose that in (1) we set $\mathcal{M} = \omega^{-1}\mathcal{H}d/dx$ where $\mathcal{H}$ is a Hilbert transform on the semi-infinite range $(0, \infty)$. That is, (1) becomes the integro-differential equation $\omega^{-1}\mathcal{H}\,dy/dx + y = F$, or, written out explicitly,

$$\frac{1}{\omega\pi}\int_0^\infty \frac{y'(t)}{t-x}\,dt + y(x) = F(x) \tag{7}$$

the integral being of Cauchy principal-value form. It is important to note that (like d/dx!), the operator $\mathcal{H}$ has an inverse that is non-unique to the extent of a single arbitrary constant. Hence two subsidiary conditions are needed for (7), like (2).

The integro-differential equation (7) also supports sinusoidal waves of frequency ω as $x \to \infty$, because of the property that $\mathcal{H}\cos(\omega x) \to -\sin(\omega x)$ as $x \to \infty$. There is clearly a very strong analogy between the constant-coefficient integro-differential equation (7) and the constant-coefficient second-order ordinary differential equation (2). Incidentally, (7) can be shown to be a representation of the classical linear problem of small-amplitude water waves generated by a small disturbance to a uniform stream. This and other applications are discussed by Varley and Walker (1989).

Similarly, $\mathcal{M} = \omega^{-1}F\,\mathcal{H}d/dx$ leads to the integro-differential equation

$$\omega^{-1}\mathcal{H}\frac{dy}{dx} + \frac{y}{F(x)} = 1 \tag{8}$$

with a non-constant restoring coefficient, analogous to the differential equation (5). If $F(\infty) = 1$, the same wave-like behaviour holds as $x \to \infty$. The function $F(x)$ of interest tends to zero rapidly as $x \to 0$, and again this implies some form of irregular singularity at $x = 0$ in the general solution, with a behaviour somewhat like that given in (6). Again, the complementary function (terms in C, D) oscillates wildly as $x \to 0$, and dominates the $F(x)$-like particular integral. All solutions already satisfy $y(0) = 0$, but only the particular solution with $C = D = 0$ is physically acceptable. Having used up two degrees of freedom in that way, we can further eliminate the waves at infinity, setting $A = B = 0$, if at all, only by special choices of ω and some parameter taken from $F(x)$.

Numerical solution of (8) is much more difficult than (5), since the Fredholm-like integral equation (8) demands treatment as a boundary-value problem on the semi-infinite interval, with a full matrix inversion after truncation to a finite interval and discretisation, and cannot be converted to an initial-value problem. Hence the accuracy is very much limited, relative to the virtually-unlimited accuracy of methods such as Runge-Kutta that are available for (5).

The operator $\mathcal{M}$ in (1) need not be linear. For example (Tuck 1990), two-dimensional large-amplitude water waves generated in $x > 0$ by a broad ship stern of finite draft which lies in $x < 0$ can be described by (1), with

$$\mathcal{M} = F(x)\left[1 - \exp(-3\mathcal{H}\arcsin\frac{1}{3\omega}\frac{d}{dx})\right]. \tag{9}$$

The resulting nonlinear singular integro-differential equation reduces to (7) upon "classical" linearisation about $y = F = 1$, and to (8) upon "double-body" linearisation for small y'. The function $F(x)$ is related to the ship's hull geometry, and ω to the (reciprocal square) Froude number based on draft.

It seems likely that, in spite of its nonlinearity, (9) has similar qualitative properties to its linearised equivalents or analogues. As $x \to \infty$, we must in general expect to generate waves, and these will be nonlinear Stokes waves. Although these waves are non-sinusoidal, they still have an arbitrary amplitude and phase, and hence there will still be two arbitrary constants A, B describing the generated waves. If we seek waveless solutions, we must use up both degrees of freedom in this fundamentally second-order problem, leaving no room to satisfy any other boundary condition, at general ω. The behaviour near $x = 0$ is likely to be at least qualitatively similar to (6), and hence if we demand solutions free of evanescent waves, so setting $C = D = 0$, we shall again need not only special values of ω, but also special values of another parameter in $F(x)$.

Some attempts have been made to solve the equivalent of (9) numerically, e.g. see Tuck and Vanden-Broeck (1984), Madurasinghe (1988). This is an even more difficult task than that for the linear problem (8), and more work is needed. The task of determining a waveless solution is a very delicate one, since the wave amplitude can become very small indeed at some values of ω, without quite vanishing, and it is difficult to distinguish numerically between a very low minimum and a zero wave amplitude. Perhaps, indeed, it is not necessary to do so; a hull that generates an unusually small wave may be as interesting as one that generates no wave at all.

REFERENCES

Madurasinghe M.A.D., "Splashless ship bows with stagnant attachment", *J. Ship Res.* **32** (1988) 194-202.

Tuck E.O., "Ship-hydrodynamic free-surface problems without waves", *The Georg Weinblum Memorial Lecture*, Berlin, November 1990.

Tuck E.O and Vanden-Broeck J.-M., "Splashless bow flows in two dimensions?", 15th Symp. Nav. Hydro., Hamburg 1984.

Varley E. and Walker J.D.A., "A method for solving singular integrodifferential equations", *I.M.A. J. App.Math.* **43** (1989) 11-45.

Evans: Is it possible to make some analytic progress with your equation (7) using Wiener-Wopf methods as Varley or Walker do, but for general $F(x)$?

Tuck: It is very likely that this is true for (7), but not for the variable-coefficient equation (8), and certainly not for the nonlinear equation (9). Even for (7), I would expect that the Varley-Walker method would lead to a triple numerical quadrature (after all, their results for $F(x) = 0$ are already double quadratures). Hence, I am not convinced that the Varley-Walker procedure is of practical value compared to direct numerical solution of the original integral equations. However, the latter does have its difficulties, associated with spurious small waves that must be filtered out (cf. Lonquet Higgens & Cokelet), and thus the Varley-Walker method may be worth further investigation for (7). Personally, I am much more interested in (8) and (9), though.

Tulin: Isn't it true that the two dimensional wavemaker problem can be presented in terms of an ordinary differential equation in the complex domain, at least to some higher order of approximation? In which case, can your discussion of waveless solutions to ODE's be applied?

Tuck: I do not know the answer to this question. However, it is possible that this can be done only for "classical" wavemakers, *i.e.*, vertical moving planes. My "$F(x)$" in some way represents a very general family of "wavemakers", with structure in both spatial directions, and I have doubts as to whether the problem can then be converted (exactly) to a differential equation. On the other hand, a few years ago I in fact used the method that you describe, and it is associated with the approximation $\sin\theta \approx 1/3 \sin 3\theta$. Indeed, subject to this approximation, the answers to your questions are yes, with the inertial operator M a complex first-order differential operator.