Hydrodynamic Impact of the Water Wave

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Wave impact is one of the major causes of structure failures in the ocean and coastal engineering. Cooker and Peregrine (1990) recently developed a mathematical model to predict the impact pressure based on the assumption of incompressible flow. Since the velocity of fluid experiences a finite change in an extremely short period during the impact, one would intend to expect that the compressibility of the fluid plays an important role. This leads to the investigation of this work.

We assume that the flow is inviscid. The governing equation may be written as

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\, \nabla . \vec{u} \right) = -\nabla p \tag{1a}$$

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho \vec{u}) = 0 \tag{1b}$$

where ρ is the density, u is the velocity and p is the pressure. Integrating equation (1a) with respect to time t and neglecting the contribution of the second term on the right hand side because the impact period is extremely short, we have

$$\rho(t)\, \vec{u}(t) - \rho(t_b)\, \vec{u}(t_b) - \int_{t_b}^{t} \vec{u}\, \frac{\partial \rho}{\partial t}\, dt = -\nabla \int_{t_b}^{t} p\, dt$$

where t_b is the start time of the impact which ends at $t=t_a$. We notice the integration on the right hand side of the equation can be also neglected upon using equation (1b). Thus

$$\rho(t)\, \vec{u}(t) - \rho(t_b)\, \vec{u}(t_b) = -\nabla P \tag{2}$$

where

$$P = \int_{t_b}^{t} p\, dt \tag{3}$$

The definition of P here is similar to that by Cooker and Peregrine. The only difference is the upper limit of the integration here is t instead of t_a and P is therefore a function of time.

Before the impact it can be assumed that the density is a constant ρ_0 and correspondingly

$$\nabla.\vec{u}(t_b) = 0$$

Substituting this and equation (2) into (1b), we have

$$\frac{\partial \rho}{\partial t} - \nabla^2 P(t) = 0$$

Using

$$\frac{\partial \rho}{\partial t} = \frac{1}{c^2}\frac{\partial p}{\partial t} = \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} \qquad (4)$$

where c is the speed of the sound, we obtain the governing equation for P

$$\frac{\partial^2 P}{\partial t^2} = c^2\nabla^2 P \qquad (5)$$

This is the well known wave equation as we expected to obtain.

We now consider the two dimensional example solved by Cooker and Peregrine based on incompressible flow: wave impact on a vertical wall. We define the Cartesian coordinate system 0-xy so that x is on the free surface and y is on the wall. The boundary conditions may be written as

$$P = 0 \qquad \text{on } y=0 \qquad (6a)$$

$$\frac{\partial P}{\partial y} = 0 \qquad \text{on } y=h \qquad (6b)$$

$$P \to 0 \qquad \text{as } x \to \infty \qquad (6c)$$

where h is the water depth. On the wall, we assume that the horizontal velocity as

$$u(t) = -U_0 T(t) \qquad (7)$$

within the impact zone $-b<y<0$ as named by Cooker and Peregrine. Here T(t) is a fucntion reflecting the change of the velocity with $T(t_b)=1$ and $T(t_a)=0$, which allows the flexibility of the wall and any air possibly trapped during the impact. Using equations (2) and (4), we have

$$\frac{\partial P}{\partial x} = -U_0 \; [\; \rho(t)\; T(t) \; - \; \rho_0]$$

$$= -U_0 \; \{ \frac{1}{c^2}\frac{\partial P}{\partial t} T(t) + \rho_0[T(t)-1]\} \qquad \text{on } x=0, \quad -b<y<0 \qquad (6d)$$

In general, the solution of above equation can be written as

$$P(x,y) = \sum_{n=1}^{\infty} f_n(x,t)\sin(\lambda_n y) \tag{8}$$

where

$$\frac{\partial^2 f_n}{\partial t^2} = c^2 \frac{\partial^2 f_n}{\partial x^2} - c^2\lambda_n^2 f_n \tag{9a}$$

$$c^2 \frac{\partial f_n}{\partial x} + U_0 \frac{\partial f_n}{\partial t} T(t) = -\frac{2\rho_0 U_0}{\lambda_n h} c^2 [T(t)-1]\,[1-\cos(\lambda_n b)], \quad \text{on } x=0 \tag{9b}$$

$$\lambda_n = (n + \frac{1}{2})\frac{\pi}{h} \tag{9c}$$

and $f_n \to 0$, as $x \to \infty$. We know that

$$\frac{\partial f_n}{\partial t} = O(c\frac{\partial f_n}{\partial x})$$

Thus the second term of the left hand side of equation (9b) can be neglected if $U_0 \ll c$ which is the case in many hydrodynamic problems. Equation (9b) may be approximated as

$$\frac{\partial f_n}{\partial x} = V(t), \qquad \text{on } x=0 \tag{10}$$

which is exact if $T(t)=0$, where $V(t)$ is the right hand side of equation (9b) without c^2.

Upon performing the Laplace transform of equation (9a) with respect to time, we obtain

$$f^*(x,s) = -V(s)\exp\{-x\sqrt{[(s/c)^2+\lambda_n^2]}\}/[(s/c)^2+\lambda_n^2]$$

where the initial conditions $f(x,t_b)=\partial f(x,t_b)/\partial t=0$ with $t_b=0$ and the boundary conditions in equation (10) and at infinity have been used. Thus with r being a positive real

$$f_n = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} A(s)\exp\{-x\sqrt{[(s/c)^2+\lambda_n^2]}+st\}\,ds$$

$$= \frac{-1}{2\pi i}\int_0^t V(t-\tau)\int_{r-i\infty}^{r+i\infty}\frac{\exp\{-x\sqrt{[(s/c)^2+\lambda_n^2]}+s\tau\}}{\sqrt{[(s/c)^2+\lambda_n^2]}}\,ds\,d\tau$$

$$= -c\int_{x/c}^{t} V(t-\tau)\,J_0[\lambda_n\sqrt{(c^2\tau^2-x^2)}]\,d\tau \tag{11}$$

where the last equation is obtained by using the result in Erdelyi

(1954, Vol.1, pp.248). When $c \to \infty$ and $T(t)=0$ for $t>0$, the above equation reduces to

$$f_n = -2\rho_0 U_0 [1-\cos(\lambda_n b)]/(\lambda_n h) \int_0^\infty \frac{\tau}{\sqrt{(\tau^2+x^2)}} J_0(\lambda_n \tau)\, d\tau$$

$$= -2\rho_0 U_0 [1-\cos(\lambda_n b)]/(\lambda_n^2 h) \exp(-\lambda_n x)$$

as obtained by Cooker and Peregrine for the incompressible flow, where eq.6.554 of Gradshteyn and Ryzhik (1963) has been used.

Our major concern here is the impact pressure on the wall. Cooker and Peregrine used an emperical equation $p_{pk}=2P/\Delta t$ to calculate the peak pressure p_{pk}, since in their formulation P is not a function of time. By differentiating equation (3) and making use of (8) and (11), we are not only able to calculate the peak pressure but also the history of the impact pressure. Further discussions will be presented in the workshop.

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References

1. Cooker, M.J. and Peregrine, D.H. (1990) "A Model of the shock pressure from breaking waves", 5th Int. Workshop on Water waves and Floating Bodies, Manchester, England.

2. Erdelyi, A. (1954) "Tables of Integral Transforms", McGraw-Hill Book Company.

3. Gradshteyn, I.S. and Ryzhik, I.M. (1963) "Tables of integrals series and products", Acadamic Press.

Yue: I understand from your abstract that c is the 'normal' speed of sound in water (and you mentioned $c = 1500$ m/s in your presentation) rather than some reduced speed due to the presence of bubbles. Furthermore, you spoke of the problem of Cooker & Peregrine (1990) as a motivation for this work. I find it hard to imagine that for breaking wave impact problems, acoustic pressures associated with c can be involved. Would you comment on this?

Wu: We hydrodynamicists have become used to Laplace's equation, and there is a tendency to take for granted what we have neglected. One of the conclusions of incompressible flow is that any disturbance in the fluid will be immediately felt by the entire fluid domain. In reality, the disturbance is transmitted by compressing the fluid and travels at the speed of sound c. Thus the effect of c is always important near the front of acoustic waves. For hydrodynamic problems, we are mainly concerned with a region surrounding the disturbance and the effect of c can be neglected. In the inner region, it will only be a matter of seconds before we can we can assume that the entire fluid domain has felt the disturbance. For impact problems, on the other hand, with a time scale measured in milliseconds, the disturbance will only have traveled a few meters. Thus we expect that for some geometries, the effect of c will be important. My work was initially motivated by that of Peregrine & Cooker but Equation (15) is for general purpose. The case considered by Peregrine & Cooker is used as an example to support the above argument. For this particular case, the effect of c on the peak pressure is important and its effect on $\int_0^{t_a} p\,dt$ will diminish like $O(\frac{1}{\sqrt{(ct_a)^3}})$ if ct_a is large.

Cooker: It is interesting to include compressibility but I think one must consider the change of c with distance from the wall. Near the wall a bubble concentration of 50% can make $c = 20$ m/s, while $c = 1500$ m/s far away. Shock waves/sound waves will strongly refract in such a non-uniform medium.

Wu: As mentioned in reply to Yue's question, equation (15) is for general purpose. The problem of impact on a vertical wall is treated as an example. For this particular case, the variation of c may also be important. My paper is nevertheless to draw the attention of hydrodynamicists to the seemingly forgotten effect of c. If c is only 20 m/s in some cases, its effect will be even more important, which further supports my argument.

Palm: The left-hand side in your equation of sound is very large. What about the right-hand side: c is very large, can you see anything about the order of the ∇^2 operation? (i.e. the horizontal length scale)

Wu: We may write equation (5) as $\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} = \nabla^2 P$. Here the very large c^2 will be balanced by very large $\frac{\partial^2 P}{\partial t^2}$ which is due to P experiencing a finite change within an extremely short period. To investigate the importance of the effect of C, for a particular case I agree that we should relate the length scale ct with that of the body and of the fluid domain as I discussed in my reply to Yue.

Tulin: (i) I do remember that acoustic pressures (ρcu) were measured on flat nosed bodies impacting on flat wakes (laboratory), but they were of extremely small duration with negligible impulse. (ii) In modeling wave impact, I would worry about the presence of clouds of small bubbles in the zone of high pressure not far beneath the free surface. The high pressures seem localized, and within the order of milliseconds a cloud of small bubbles could collapse and further increase the pressure there.

Wu: (i) It is difficult to comment without knowing how the experiment was conducted and what was really measured. But my stubborn intuition tells me that the compressibility will be important for a flat plate entering the water at large speed. (ii) Indeed, not only the bubbles will occur, but water will splash in all directions. This is certainly not a topic of continuum mechanics. How can we include these effects? The answer cannot be easier, I do not know!