ON THE SECOND ORDER WAVE DIFFRACTION PROBLEMS

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Except several existing numerical solutions of the second order diffraction and radiation problems which are based on the application of Green's identity using the linear wave source potential as Green function, there exist other alternative methods which do not require the evaluation of the infinite free surface integrals. An alternative methodology was developed by Sclavounos(1988). In his method, the "difference- and sum-frequency Green Function" obtained from the solutions of initial-value problems that ensure they satisfy the proper radiation condition at infinite have been introduced. Another was developed by Wu (1988). But the numerical evaluation of the second wavw order forces is still very difficult with the above methods. Here, we will give an definition and several alternative expressions of three dimensional diffraction and radiation function in finite depth water, an radiation condition of the second order diffraction potential also derived by examine the behaviours of the second order solution at far field. The present method may be more convenient for computation than others.

Let us consider the interaction of random ambient waves with a fiexd body in finite depth sea and approximate a random seaway by the linear surperposition of a sufficiently large number of reqular plane progres -sive wave components of different frequencies $\{\omega_n\}$ and headings $\{A_n\}$. By denoting the linear order incident and scattering potential by $Re\{\varphi_m^I(\mathbf{p})\exp(-i\omega_m t)\}$ and $Re\{\varphi_m^S(\mathbf{p})\exp(-i\omega_m t)\}$ respectively, one can find the second order scattering potential ϕ_2^S has following expression:

$$\phi_2^S(P,t)=\sum_{m,n=1} Re\{\phi_{nm}^+(\mathbf{p})\exp(-i\Omega^+ t)+\varphi_{nm}^-(\mathbf{p})\exp(-i\Omega^- t)\} \qquad \Omega^{\pm}=\omega_n\pm\omega_m \qquad (1)$$

where, The sum- and difference- frequency potentials $\phi_{nm}^+(\mathbf{p})$ and $\phi_{nm}^-(\mathbf{p})$ satisfy Laplace equation in the fluid domain D and boundary conditions:

$$(\partial_3-\Lambda^{\pm})\phi_{nm}^{\pm}=P_{nm}^{\pm}(\mathbf{p}),\ \text{on}\ x_3=0;\ \partial_3\phi_{nm}^{\pm}=0,\ \text{on}\ x_3=-H;\ \nabla\phi_{nm}^{\pm}\Rightarrow 0,\ \text{as}\ r\Rightarrow\infty \qquad (2A)$$

$$\mathbf{n}\nabla(\phi_{nm}^{\pm}+\varphi_{nm}^{\pm})=0,\ \text{on}\ \mathbf{p}\in S_B \qquad (2B)$$

here, $\Lambda^{\pm}=[(\Omega^{\pm})^2+i\mu\Omega^{\pm}]/g$, $\partial_j=\partial/\partial x_j\ (j=1,2,3)$, $\mathbf{p}=(x_1,x_2,x_3)$, $r=\sqrt{x_1^2+x_2^2}$, sea bottom depth H is constant, $\mathbf{n}$ is the unit normal vector pointing into the body and S_B the wetted surface of body. x_3 axis is the vertical axis, positive upward, and $x_3=0$ corresponds to the mean free surface. φ_{nm}^+ and φ_{nm}^- express the second order sum- and difference- incident potential, respectively. The forcing term $P_{nm}^{\pm}(\mathbf{p})$ can be expressed as

$$P^{\pm}(P)=\frac{i}{2g}[\Omega^{\pm}\nabla\varphi_n^S\nabla\varphi_m^{S,\pm}-\omega_n\varphi^S L_m\varphi^{S,\pm}+2\Omega^{\pm}\nabla\varphi^S\nabla\varphi^{I,\pm}\omega_n\varphi_n^S L_m\varphi^{I,\pm}\mp\omega_m\varphi_m^{I,\pm}L_n\varphi^S] \qquad (3)$$

where $L_m=\partial_3^2-\nu_m\partial_3$, $\nu_m=\omega_m^2/g$, $\varphi_m^{I,+}=\varphi_m^I=(\varphi_m^{I,-})^*$, and $\varphi_m^{S,+}=\varphi_m^S=(\varphi_m^{S,-})^*$. The parameter μ is the Rayleigh stress which ensures φ_m^S and $\phi_{nm}^{\pm}$ satisfy a proper radiation condition at far field(Wu, 1988).

Noticing that the linear incident potential can be difined by

$$\varphi_n^I(x)= igA_n[\omega_n \mathrm{ch}k_nH]^{-1}\mathrm{ch}k_n(x_3+H)\exp(i\mathbf{k}_n\mathbf{x}) \qquad (4)$$

where, $\mathbf{x}=(x_1,x_2)$. $\mathbf{k}=k_n(\cos\beta_n,\sin\beta_n)$, the wavemunber k_n is the positive root of the dispersion equation:

$$F(k,\nu_n)=\nu_n-k\tanh kH=0 \qquad (4A)$$

and, in connection with the source distribution method, the solution of the linear scattering potential φ_n^S can be expressed as

$$\varphi_n^{S,\pm}(\mathbf{p})= \int_{S_B}\sigma_n^{\pm}(\mathbf{q})G(\mathbf{p},\mathbf{q};\nu_n^{\pm})dS, \qquad \mathbf{q}=(q_1,q_2,q_3)\in S_B \quad \mu=0^+ \qquad (5)$$

where, $\nu_n^{\pm}=(\omega_n^2\pm i\mu\omega_n)/g$, σ_B is the source density, $\sigma_B=(\sigma_B^-)^*=\sigma_B^+$, and the Green function G is the solution of below boundary value problem:

$$\nabla^2G(\mathbf{p},\mathbf{q};\nu_n^{\pm})=\delta(\mathbf{p}-\mathbf{q}), \qquad \mathbf{p},\mathbf{q}\in D \qquad (6)$$

$$(\partial_3-\nu_n^{\pm})G=0, \text{ on } x_3=0;\ \partial_3G=0, \text{ on } x_3=-H;\ \nabla G \Rightarrow 0, \text{ as } r\Rightarrow\infty \qquad (6A)$$

one can easily find that $\phi_{nm}^{\pm}$ has a special solution which is governed by Laplace equation and boundary condition(2A) and can be written as

$$\phi_{\mathbf{p},nm}^{\pm}(\mathbf{p})=\int_{S_B}dS_q\sigma_n(\mathbf{q})L_{nm}^{D,\pm}[G_D(\mathbf{p},\mathbf{q};\Lambda^{\pm},\nu_n^+,\pm\mathbf{k}_m)]+ \int_{S_B}dS_q\int_{S_B}dS_{q'}\sigma_n(\mathbf{q})\sigma_m^{\pm}(\mathbf{q}')$$

$$.L_{nm}^{S,\pm}[G_S(\mathbf{p},\mathbf{q},\mathbf{q}';\Lambda^{\pm},\nu_n,\nu_m^{\pm})] \qquad \mathbf{q}'=(q_1',q_2',q_3')\in S_B \quad \mu=0^+ \qquad (7)$$

where, the differential operetor $L_{nm}^{D,\pm}$ and $L_{nm}^{S,\pm}$ are defined by

$$L_{nm}^{D,\pm}=\frac{-1}{2\omega_m}A_m[2\Omega^{\pm}(\nu_n\nu_m\pm ik_m\nabla_1)-\omega_n(k_m^2-\nu_m^2)\mp\omega_m(\partial^2/\partial q_3^2-\nu_n^2)] \qquad (7A)$$

$$L_{nm}^{S,\pm}=\frac{i}{2g}[\Omega^{\pm}(\nu_n\nu_m+\nabla_1\nabla_1')-\omega_n(\partial^2/\partial q_3'^2-\nu_m^2)] \qquad (7B)$$

where, $\nabla_1=(\partial/\partial q_1,\partial/\partial q_2)$, $\nabla_1'=(\partial/\partial q_1',\partial/\partial q_2')$, and the diffraction and scattering Green function G_D and G_S are defined by following boundary value problems:

$$\nabla^2G_{D,S}=0,\ \mathbf{p}\in D; \qquad \partial_3G_{D,S}=0, \text{ on } x_3=-H; \quad G_{D,S}\Rightarrow 0, \text{ as } r\Rightarrow\infty;$$

$$(\partial_3-\Lambda^{\pm})G_D=G(\mathbf{p},\mathbf{q};\nu_n)\exp(\pm i\mathbf{k}_m\mathbf{x}),\ (\partial_3-\Lambda^{\pm})G_S=G(\mathbf{p},\mathbf{q};\nu_n)G(\mathbf{p},\mathbf{q}';\nu_m^{\pm}) \text{ on } x_3=0 \qquad (8)$$

Assume that the function G_D and G_S decay sufficiently rapidly as $r\Rightarrow\infty$ so that their Fourier transforms with respect to (x_1,x_2) coordinates exsit. By using Fourier transformation method, One can obtained that

$$G_D(\mathbf{p},\mathbf{q};\Lambda^{\pm},\nu_n^+,\pm\mathbf{k}_m)=\frac{-1}{4\pi^2}\iint_{-\infty}^{\infty}du_1du_2\ F(\mathbf{u},x_3,\Lambda^{\pm})F(\mathbf{u}-\mathbf{k}_m,q_3,\nu_n^+)$$

$$.\exp[\pm i\mathbf{u}(\mathbf{x}-\mathbf{x}_1)\pm i\mathbf{k}_m\mathbf{x}_1] \qquad (9)$$

$$G_S(\mathbf{p},\mathbf{q},\mathbf{q}';\Lambda^{\pm},\nu_n,\nu_m^{\pm})=\frac{-1}{(2\pi)^4}\iiiint_{-\infty}^{\infty}du_1du_2dv_1dv_2\ F(\mathbf{u},x_3,\Lambda^{\pm})F(\mathbf{v},q_3,\nu_n^+)$$

$$.F(\mathbf{u}-\mathbf{v},q_3',\nu_m^{\pm})\exp[i\mathbf{u}(\mathbf{x}-\mathbf{x}_2)+i\mathbf{v}(\mathbf{x}_2-\mathbf{x}_1)] \qquad (10)$$

where $\mathbf{v}=(v_1,v_2), v=\sqrt{v_1^2+v_2^2}$, $\mathbf{u}=(u_1,u_2), u=\sqrt{u_1^2+u_2^2}$, $\mathbf{x}_1=(q_1,q_2)$, $\mathbf{x}_2=(q_1',q_2')$ and $F(u,z,\sigma)=\mathrm{ch}u(z+H)/(\sigma \mathrm{ch}uh-u\mathrm{sh}uH)$. By changing the Cartesian to polar coordinate system, alternative forms for (9) and (10) which may be more convenient for computations are obtained as

$$G_D(\mathbf{p},\mathbf{q};\Lambda^{\pm},\nu_n^+,\pm k_m)=\frac{-1}{4\pi^2}\int_0^{2\pi}d\theta\int_0^{\infty}F(\rho^{\mp}(u,k_m,\theta-\beta_m+\theta_1),q_3,\nu_n^+) \cdot \exp[iR_1 u\cos\theta \pm ik_m x_1]u\,du \tag{11}$$

$$G_S(\mathbf{p},\mathbf{q},\mathbf{q}';\Lambda^{\pm},\nu_n,\nu_m^{\pm})=\frac{-1}{8\pi^3}\int_0^{2\pi}d\theta\int_0^{\infty}du\int_0^{\infty}uvF(u,q_3',\nu_m^{\pm})F(v,q_3,\nu_n^+) \cdot F(\rho^+(u,v,\theta),x_3,\Lambda^{\pm})J_0(\rho^+(R_2u,R_1v,\theta-\theta_{21}))dv \tag{12}$$

where, $R_j(\cos\theta_j,\sin\theta_j)=\mathbf{x}-\mathbf{x}_j\ (j=1,2)$, $\rho^{\pm}(u,v,\theta)=[u^2+v^2\pm 2uv\cos\theta]^{1/2}$, $J_0(z)$ is the zero-order Bessel function, $\theta_{21}=\theta_2-\theta_1$. Noticing that the inner integral in formula (9) can be treated as a complex integral along the real axis in the complex plane u_1 and this path of integration can be modified by introducing a closed integration contour comprising the real axis, we obtained another alternative form for G_D which may be more convenient for computation than both expressions (9) and (11) as

$$G_D(\mathbf{p},\mathbf{q};\Lambda^{\pm},\nu_n^+,\pm k_m)=\frac{-1}{2\pi}\sum_{j=1}^{\infty}\{C(x_3,\Lambda^{\pm},\lambda_j)\int_{-\infty}^{\infty+i\gamma_j}\exp[iR_1\lambda_j \mathrm{ch}(t-\varepsilon_j)\pm ik_m x_1] \cdot F(\rho^{\mp}(\lambda_j,k_m,t_{mj}),q_3,\nu_n)dt + C(q_3,\nu_n^+,k_{nj})\int_{-\infty}^{\infty+i\gamma_j'}dt \; \exp[iR_1k_{nj}\mathrm{ch}(t-\varepsilon_j')\pm ik_m x]F(\rho^{\pm}(k_{nj},k_m,t_{mj}'),x_3,\Lambda^{\pm})\} \tag{13}$$

where $t_{mj}=\beta_m-\theta_1+\varepsilon_j+it$, $t_{mj}'=\beta_m-\theta_1+\varepsilon_j'+it$, $\varepsilon_{j(\geq 2)}=\varepsilon_j'=\gamma_j=\gamma_j'=0$, $\varepsilon_1'=\mathrm{sign}(\Omega^{\pm})\frac{\pi}{2}$, $\varepsilon_1=\frac{\pi}{2}$, $\gamma_1=\pi$, $\gamma_1'=\mathrm{sign}(\Omega^{\pm})\pi$, $\{k_{nj}\}$ and $\{\lambda_j\}$ are the complex roots of the dispersion equations: $F(\lambda,\nu_n^+)=0$ and $F(\lambda,\Lambda^{\pm})=0$ in the upper half complex plane of λ. Specially, $k_{n1}=k_n$ and $\mathrm{Im}(\lambda_1)=0^+$, as $\mu=0^+$. The integration contours are shown in figure 1 and

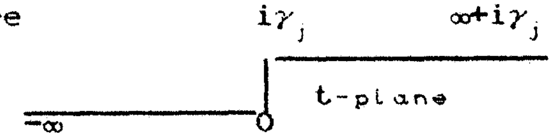

Fig. 1. The integration contour in the complex t-plane

$$C(z,\nu,k)=k^2[(\nu^2-k^2)H-\nu]^{-1}\mathrm{ch}k(z+H)/\mathrm{ch}kH; \tag{14}$$

Similar to G_D, another alternative form for G_S is

$$G_S(\mathbf{p},\mathbf{q},\mathbf{q}';\Lambda^{\pm},\nu_n,\nu_m^{\pm}) \tag{15}$$
$$=\frac{-1}{4\pi^2}\sum_{p=1}\sum_{j=1}\{C(q_3,\nu_n^+,k_{np})C(x_3,\Lambda^{\pm},\lambda_j)U_{jp}^+(R_2,\eta,q_3';\theta_{20},0;\lambda_j,k_{np},\nu_m^{\pm}) + C(x_3,\Lambda^{\pm},\lambda_j)C(q_3',\nu_m^{\pm},k_{mp})U_{jp}^-(R_1,\eta,q_3;\theta_{10},0;\lambda_j,k_{mp},\nu_n) + C(q_3,\nu_n^+,k_{np})C(q_3',\nu_m^{\pm},k_{mj})U_{jp}^+(R_1,R_2,x_3;\theta_{10},\theta_{20};k_{np},k_{mj},\Lambda^{\pm})\}$$

where, $\mathbf{x}_1-\mathbf{x}_2=\eta(\cos\theta_0,\sin\theta_0)$, $\theta_{j0}=\theta_j-\theta_0 (j=1,2)$

$$U_{jp}^{\pm}(x,y,z;\alpha,\beta;a,b,c)=\int_{-\infty}^{\infty+i\gamma_j'}dt_1\int_{-\infty}^{\infty+i\gamma_p''}F(\rho^{\pm}(a,b,it_{1j}-it_{2p}''),z,c) \cdot \exp[iax\mathrm{ch}(t_{1j}-i\alpha)+iby\mathrm{ch}(t_{2p}''-i\beta)]dt_2 \tag{15A}$$

in the above fomula, $\alpha,\beta\in(-\frac{\pi}{2},3\frac{\pi}{2})$. $\alpha'=\beta'=0$, as $\alpha,\beta\in(-\frac{\pi}{2},\frac{\pi}{2})$, and $\alpha'=\beta'=-\pi$, as $\alpha,\beta\in(\pi/2,3\pi/2)$. $t_{1j}=t_1-i(\varepsilon_j'+\alpha')$, $t_{2p}''=t_2''-i(\varepsilon_p''+\beta')$. $\gamma_1'=f(a)\pi$, $\gamma_1''=f(b)\pi$, $\gamma_{j(\geq 2)}''=\gamma_j'=0$, and $\varepsilon_1'=f(a)\pi$, $\varepsilon_1''=f(b)\pi$, $\varepsilon_{j(\geq 2)}''=\varepsilon_j'=0$. $f(a)=\mathrm{sign}[\mathrm{Re}(a)]$.

Using the method of the stationary phase and noticing that, in the integrands of the integrals in formula (13) and (15A), the stationary phase points occur at $t=\varepsilon_j$ $t=\varepsilon_j'$, $t_{1j}=i\alpha$ and $t_{2p}''=i\beta$ respectively, we find, for large r,

$$G_D(\mathbf{p},\mathbf{q};\Lambda^\pm,\nu_n^+,\pm\mathbf{k}_m)=C(x_3,\Lambda^\pm,\lambda_1)F(\rho^\mp(\lambda_1,k_m,\alpha_m),q_3,\nu_n)(2\pi\lambda_1R_1)^{-1/2}$$
$$\exp[i(\lambda_1R_1\pm k_mx_1-\tfrac{3}{4}\pi)] \quad +C(q_3,\nu_n^+,k_{nj})F(\rho^\pm(k_n,k_m,\alpha_m),x_3,\Lambda^\pm)$$
$$(2\pi k_nR_1)^{-1/2}\exp[i(k_nR_1\pm ik_mx-\tfrac{3}{4}\pi)]+O(r^{-3/2}) \qquad (16)$$

$$G_S(\mathbf{p},\mathbf{q},\mathbf{q}';\Lambda^\pm,\nu_n,\nu_m^\pm)= C(x_3,\Lambda^\pm,\lambda_1)(2\pi\lambda_1r)^{-1/2}G_D(\mathbf{q}',\mathbf{q}'';\nu_n^+,\nu_m^\pm,-\lambda_1\mathbf{e})$$
$$.\exp[i(\lambda_1r\pm k_mx_1-\tfrac{3}{4}\pi)]+\frac{1}{2\pi}F(\rho^\pm(k_n,k_m,0),x_3,\Lambda^\pm)(k_nk_mR_1R_2)^{-1/2}$$
$$.C(q_3,\nu_n^+,k_n)C(q_3',\nu_m^\pm,k_m)\exp[i(k_nR_1-\tfrac{\pi}{4}\pm k_mR_2\mp\tfrac{\pi}{4})]+O(r^{-3/2}) \qquad (17)$$

where $\mathbf{e}=(\cos\theta_0,\cos\theta_0)$, $\alpha_m=\beta_m-\alpha$, $r\cos\alpha=x_1$.

From the above formulas, we can write the radiation condition of the second order diffraction potential as below

$$L_r\phi_{nm}^\pm= C_{nm}^{D,\pm}F(\rho^\pm(k_n,k_m,\alpha_m),x_3,\Lambda^\pm)\psi_n^+(\mathbf{x})\varphi_{m0}^I(\mathbf{x})+O(r^{-1}) \qquad (18)$$

or

$$L_r\phi_{nm}^\pm= C_{nm}^{D,\pm}F(\rho^\pm(k_n,k_m,\alpha_m),x_3,\Lambda^\pm)\psi_n^+(\mathbf{x})\varphi_{m0}^I(\mathbf{x})$$
$$+C_{nm}^{S,\pm}F(\rho^\pm(k_n,k_m,0),x_3,\Lambda^\pm)\psi_n^+(\mathbf{x})\psi_m^\pm(\mathbf{x})+O(r^{-3/2}) \qquad (18A)$$

where $L_r=\partial/\partial r-i\lambda_1$, $\varphi_{m0}^I(\mathbf{x})=\varphi_m^I(\mathbf{p})|x_3=0$, and $\psi_m^\pm(\mathbf{x})=\rho_n^{S,\pm}(\mathbf{p})|x_3=0$.

$$C_{nm}^{D,\pm}=\frac{-1}{2g}(k_n\pm k_m\cos\alpha_m-\lambda_1)[2\Omega^\pm(\nu_n\nu_m\mp k_mk_n\cos\alpha_m)-\omega_n(k_m^2-\nu_m^2)\mp\omega_m(k_n^2-\nu_n^2)]$$
$$C_{nm}^{S,\pm}=\frac{i}{2g}(k_n\pm k_m-\lambda_1)\ [\Omega^\pm(\nu_n\nu_m\mp k_nk_m)-\omega_n(k_m^2-\nu_m^2)] \qquad (20)$$

We can see from formula (16) and (17) that there exist two typical behaviours of $\phi_{nm}^\pm$ at far field, one is the locked phase presure wave systems similar to the far field behaviours of the free surface forcing term $P_{nm}^\pm(\mathbf{p})$, the another is the free scattering wave systems with frequency $\omega_n\pm\omega_m$. All the second order diffraction wave systems are implied in the radiation condition (18) or (18A), the radiation condition (18A) may be more suited for numerical evaluation than (18) when the finite elememt method or the boundary element method using the Rankine source as the Green function is used.

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