Trapped modes about thin vertical plates in a wave tank

D V Evans
School of Mathematics
University of Bristol,BS8 1TW, UK

Introduction

At the last workshop a proof was given [1] for the existence of trapped modes in a wave tank containing a symmetrically-placed vertical circular cylinder extending throughout the depth of the tank, provided that the cylinder was sufficiently small. Computations suggested that the modes, antisymmetric about the centre-plane of the channel and symmetric about a vertical plane through the centre of the cylinder perpendicular to the channel walls, appeared to exist for all sizes of cylinder even when completely blocking the channel. A full version of this work can be found in [2].

In the present paper we extend our understanding of trapped modes by proving their existence for a different geometry, namely a vertical thin plate on the centre-line of the wave tank and extending throughout the water depth. We show that modes antisymetric about the centre-line exist provided the plate is wide enough and that as the width increases more and more trapped modes arise. The technique, due originally to Mittra [3] and adapted by Jones [4] provides a constructive proof which enables the trapped modes to be computed extremely easily. The method depends upon being able to solve explicitly a related problem for a semi-infinite plate.

Formulation

The depth dependence cosh $k(y+h)$ can be extracted leaving a two-dimensional problem for a function $\phi(x,y)$;

$$(\nabla^2 + k^2)\phi = 0 \qquad 0 \le y \le 1 \quad \text{all } x \tag{2.1}$$

$$\phi = 0 \qquad y = 0, \quad x > a \tag{2.2}$$

$$\phi_y = 0 \qquad \begin{cases} y = 0, \ 0 < x < a & (2.3) \\ y = 1, \ x > 0 & (2.4) \end{cases}$$

$$\phi \longrightarrow 0 \qquad x \longrightarrow \infty, \ 0 \le y \le 1 \tag{2.5}$$

$$\phi_r = 0(r^{-\frac{1}{2}}), \qquad r = \{(x-a)^2+y^2\}^{\frac{1}{2}} \longrightarrow 0 \tag{2.6}$$

$$\phi_x = 0 \qquad x = 0, \ 0 < y < 1 \tag{2.7}$$

Here k is the real positive solution of

$$\omega^2 = gk \tanh kh. \tag{2.8}$$

Notice that the channel has half-width unity and depth h, and the half-width of the plate is a, and because of antisymmetry about $y = 0$ and assumed symmetry about $x = 0$ we need only consider $x \ge 0$, $0 \le y \le 1$.

We shall prove the existence of a non-trivial solution of (2.1 - (2.7) for some value of $k < \frac{\pi}{2}$ and a sufficiently large value of a. Then the trapped mode frequency ω is given by (2.8)with this value of k.

We denote $0 \le y \le 1$, $0 \le x \le a$ by region I; $0 \le y \le 1$, $x \ge a$ by region II and their common boundary $0 \le y \le 1$, $x = a$, by L. In region I we write

$$\phi(x,y) = \sum_{n=0}^{\infty} U_n^{(1)} \frac{\cosh k_n x}{k_n \sinh k_n a} \psi_n(y) \tag{2.9}$$

where $\psi_n(y) = (\epsilon_n)^{\frac{1}{2}}\cos p_n y$, $p_n = n\pi$, $n = 0,1,2,\ldots$, $\epsilon_o = 1$, $e_n = 2$, $n > 0$, $k_n = (p_n{}^2-k^2)^{\frac{1}{2}}$, $k_o = ik$ and $\phi(x,y)$ satisfies (2.1), (2.3), (2.4) and (2.7). In region II we write

$$\phi(x,y) = \sum_{n=1}^{\infty} U_n^{(2)} (-\kappa_n)^{-1} e^{-\kappa_n(x-a)} \Psi_n(y) \tag{2.10}$$

where $\Psi_n(y) = 2^{\frac{1}{2}}\sin\ell_n y$, $\ell_n = (n-\frac{1}{2})\pi$, $n = 1,2,,\ldots$, $\kappa_n = (\ell_n{}^2-k^2)^{\frac{1}{2}} > 0 \ \forall\ n$. Matching ϕ, ϕ_x across L gives

$$\sum_{n=0}^{\infty} U_n^{(1)}\psi_n(y) = \sum_{n=1}^{\infty} U_n^{(2)}\Psi_n(y) \qquad y \in L \tag{2.11}$$

and
$$\sum_{n=0}^{\infty} U_n^{(1)}k_n^{-1} \coth k_n a\, \psi_n(y) = \sum_{n=1}^{\infty} U_n^{(2)}(-\kappa_n)^{-1}\, \Psi_n(y) \qquad y \in L. \tag{2.12}$$

If we now multiply (2.11) and (2.12) by $\psi_m(y)$ and integrate over L, and then eliminate $U_m{}^{(1)}$ between the resulting infinite system of equations we obtain

$$\sum_{n=1}^{\infty} U_n\left\{\frac{1}{\kappa_n-k_m} + \frac{\xi_m}{\kappa_n+k_m}\right\} = 0, \quad (m = 0,1,2,\ldots) \tag{2.13}$$

where $U_n = U_n^{(2)}\ell_n/2^{\frac{1}{2}}k_o\kappa_n$ and $\xi_m = e^{-2k_m a}$ and we seek a solution of (2.13) satisfying $U_n = 0(n^{-\frac{1}{2}})$ as $n \longrightarrow \infty$ to ensure that (2.6) is satisfied.

Before proving that (2.13) has a solution consider the following heuristic argument. For $a >> 1$ then away from the edge $x = a$, $\phi \sim A\cos kx$. Close to the edge we appear to have a wave incident from $x = -\infty$ being totally reflected, since $k < \frac{\pi}{2}$, with reflection coefficient $R = -e^{2i\beta}$, say, with $|R| = 1$.

Thus the solution looks like $\phi \sim B(e^{-ik(x-a)}+ Re^{-ik(x-a)})$ and matching these two solutions away from $x = 0$ requires $B = \frac{1}{2}A\exp(ika)$ and

$$R = -e^{2i\beta} = e^{2ika} \tag{2.14}$$

and it is this condition which provides an approximation to the trapped mode frequencies. In fact R can be determined explicitly from the Wiener-Hopf technique whence it is found that

$$\beta = \arg \prod_{n=1}^{\infty}\frac{(1+ik/\kappa_n)}{(1+ik/k_n)} = \sum_{n=1}^{\infty} \{\tan^{-1}(k(\kappa_n)-\tan^{-1}(k/k_n)\}. \tag{2.15}$$

It is easy to see that (2.14) must have solutions since R is independent of a. Thus we can write (2.15) in the form

$$\frac{\pi}{2} - \beta = ka + n\pi,\ n \text{ an integer}.$$

Then by fixing k, and hence β, it is always possible to fix n and choose a such that this is satisfied. Indeed there are an infinite number of solutions for a, the difference between successive values approaching π/k as a increases.

Before returning to the trapped mode problem it is instructive to try and solve for R using matched eigenfunction expansions. The solution in II remains the same whilst in I now extended to $x = -\infty$

$$\phi(x,y) = \{e^{ik(x-a)} + Re^{-ik(x-a)}\}\psi_0(y) + \sum_{n=1}^{\infty} U_n^{(1)} k_n^{-1} e^{k_n(x-a)} \psi_n(y)$$

Repeating the matching procedure now results in

$$\sum_{n=1}^{\infty} \frac{U_n}{\kappa_n - k_m} = -\delta_{mo}, \quad (m = 0,1,2,\ldots) \tag{2.16}$$

as the infinite system needing to be solved with U_n as before. The explicit solution of (2.16) is achieved by considering

$$I_m = \frac{1}{2\pi i}\int_{C_N} \frac{f(z)}{z-k_m}\, dz, \quad m = 0,1,2,\ldots \tag{2.17}$$

where f(z) is a meromorphic function satisfying

(i) f(z) has simple poles at $z = \kappa_n$, $n = 1,2,\ldots$

(ii) f(z) has simple zeros at $z = k_n$, $n = 1,2,\ldots$

(iii) $f(z) = 0(z^{-\frac{1}{2}})$ as $|z| \to \infty$ on C_N, $\to \infty$. Here C_N is a sequence of circles, centre at the origin, radius $R_N = (N-\frac{1}{4})\pi$.

Then $I_N \to 0$ as $N \to \infty$ and,

$$\sum_{n=1}^{\infty} \frac{Res(\kappa_n)}{\kappa_n - k_m} = -f(k_o)\delta_{mo}, \quad (m = 0,1,2,\ldots)$$

where $Res(z_o)$ is the residue of f(z) at $z = z_o$. So (2.16) is satisfied if $f(k_o) = 1$ and $U_n = Res(\kappa_n)$, $n = 1,2,\ldots$. It can be shown that

$$f(z) = hg(z) = h\prod_{n=1}^{\infty}\frac{(1-z/k_n)}{(1-z/\kappa_n)} \tag{2.18}$$

satisfies all the required conditions where the constant h is given by $f(k_o) = hg(k_o) = 1$. Some further work confirms that R is given by $-e^{2i\beta}$ with β given by (2.15).

Returning to the trapped mode infinite system (2.13) we write

$$J_m = \frac{1}{2\pi i}\int_{C_N} f(z)\left\{\frac{1}{z-k_m} + \frac{\xi_m}{z+k_m}\right\} dz \tag{2.19}$$

and we write

$$f(z) = h(z)g(z) \tag{2.20}$$

with g(z) as before and

$$h(z) = 1 + \sum_{n=1}^{\infty} A_n/(z-k_n). \tag{2.21}$$

Then

$$\sum_{n=1}^{\infty} Res(\kappa_n)\left\{\frac{1}{\kappa_n - k_m} + \frac{\xi_m}{\kappa_n + k_m}\right\} + f(k_m) + \xi_m f(-k_m) = 0, \quad (m = 0,1,2,\ldots)$$

and (2.13) is satisfied provided

$$U_n = \mathrm{Res}(\kappa_n) \tag{2.22}$$

and

$$f(k_m) + \xi_m f(-k_m) = 0, \qquad (m = 0,1,2,\ldots). \tag{2.23}$$

We consider m > 0 first. Direct substitution in (2.20), (2.21) results in

$$A_m + \sum_{n=1}^{\infty} K_{mn}A_n = C_m \qquad (m = 1,2,\ldots) \tag{2.24}$$

where

$$C_m = \xi_m B_m, \; K_{mn} = C_m/(k_m+k_n) \tag{2.25}$$

and

$$B_m = k_m g(-k_m) \prod_{n=1}^{\infty} (1-k_m/\kappa_n) \Big/ \prod_{\substack{n=1 \\ n \neq m}}^{\infty} (1-k_m/k_n) \tag{2.26}$$

Now (2.24) has a unique solution A_n with $\Sigma A_n^2 < \infty$ if $\Sigma C_m^2 < \infty$ and $\Sigma\Sigma K_{mn}^2 = \rho < 1$ and because of the term $\xi_m = e^{-2k_m a}$ it is easily shown that these conditions are satisfied for sufficiently large a. Thus the A_n exist and are unique as does h(z) and hence f(z) from (2.20).

Finally when m = 0 we require from (2.23)

$$\xi_0 = e^{2ika} = \frac{-f(-ik)}{f(ik)} = \frac{-g(-ik)h(-ik)}{g(ik)h(ik)} = -e^{-2i\beta}e^{2i\delta} = Re^{2i\delta}$$

since arg g(ik) turns out to be β as given by (2.15).

Here $\delta = \arg\left\{1- \sum_{n=1}^{\infty} A_n/(k_n+ik)\right\}$. Thus the exact condition for trapped modes is

$$Re^{2i\delta} = e^{2ika} \tag{2.27}$$

or

$$\frac{\pi}{2} - \beta + \delta = ka + n\pi, \text{ n integer.} \tag{2.28}$$

As a increases $\delta \rightarrow 0$ rapidly since $A_n = 0(e^{-2k_n a})$ and (2.27) reduces to (2.14).

It is straightforward to prove that (2.28) has an infinity of solutions for a sufficiently large. Similar arguments can be used to prove the existence of trapped modes which are antisymmetric about x = 0 for a large enough.

Conclusion

It has been proved that there exist trapped modes in the vicinity of a vertical thin plate placed on the centre line of a wave-tank provided the plate width is sufficiently large. In fact numerical work indicates there is always at least one trapped mode for any value of a. The problem described here is identical to a plate in a two-dimensional acoustical wave guide since the governing equation is the Helmholtz equation. In fact such modes have been observed experimentally in wind tunnels by Parker [5]. For a recent description of the occurrence of trapped modes in acoustics see Parker [6].

It is possible to use the method described here to prove the existence of trapped modes when the plate is <u>off</u> the centre-line [7]. These modes correspond to frequencies which are embedded in the continuous spectrum.

References

[1] Evans, D.V. (1991) Trapped modes near bodies in channels. Proc. 6th Int. Workshop on Water Waves and Floating Bodies, Woods Hole,

[2] Callan, M., Linton, C.M. and Evans, D.V. (1991) Trapped modes in two-dimensional waveguides. J.Fluid Mech. **229**, 51-64

[3] Mittra, R. and Lee, S.W. (1971) Analytical techniques in the theory of guided waves. Macmillan.

[4] Jones, D.S. (1979) Methods in electromagnetic wave propagation. Oxford.

[5] Parker, R. (1966) Resonance effects in wake shedding from parallel plates: some experimental observations. J.Sound and Vibr. 4(1), 62-72.

[6] Parker, R. (1989) The excitation and consequences of acoustic resonances in enclosed fluid flow around solid bodies. Proc.Inst.Mech.Eng. **203**, 9-19.

[7] Evans, D.V., Linton, C.M. and Ursell, F. (1992) Trapped modes embedded in the continuous spectrum. *To be published in* Q.J.Mech.Appl.Meth.

DISCUSSION

MILOH: In the case where the plate is off-center you presented some graphs displaying the solution as a function b/d. An interesting limit is b/d $\rightarrow$ 1, in which the plate is merged with the upper wall. Here the potential is null and it is important to investigate the asymptotic form of this limit, i.e. the precise way it approaches zero. This limiting case was not presented in the graphs nor discussed in your talk and I wonder if you can elaborate on it a bit more.

EVANS: As the plate moves closer to the side wall the trapped mode frequency approaches the first cut off frequency for the wave tank. It ought to be possible to determine this analytically but we have not yet done this.