Far-field formulae for the second-order oscillatory force on a body, due to the first-order potential

by M. McIver

Loughborough University of Technology, Loughborough, Leicestershire, LE11 3TU, U.K.

Introduction

When a small amplitude water-wave train is incident upon a fixed body, a second-order analysis predicts that the body experiences a steady force and a force at twice the frequency of the incident wave. The calculation of the steady force requires knowledge only of the first-order potential and it may be directly expressed in terms of integrals of products of first-order quantities over the mean wetted surface of the body and the waterline. However, Maruo (1960) showed that the steady, horizontal force on a body may also be written entirely in terms of the far-field amplitude of the first-order diffracted wave. The calculation of the double frequency force is more complicated because it depends in part on the second-order potential. The contribution to this force from the first-order potential may be expressed in terms of integrals of products of first-order quantities over the mean wetted surface of the body and the waterline, in a similar form to the expressions for the steady force. However, it is demonstrated below that this part of the oscillatory force may also be written in terms of the far-field amplitude of the linear diffracted wave.

Theoretical analysis

A wave is incident from the left on a two-dimensional, fixed body which is either completely submerged or intersects the mean free surface at right angles. The fluid is assumed to have infinite depth and coordinate axes are chosen with the origin in the undisturbed free surface, the x - axis horiziontal and the z - axis pointing vertically upwards. Mei (1983) showed that the contribution to the double frequency force on the body from the first-order potential, $Re[-igA\phi_1 e^{-i\omega t}/\omega]$, is given by

$$\frac{\epsilon^2 f_{2x}^{(1)}}{\rho g A^2} = \frac{1}{4K}\int_{\overline{S}_B} (\nabla\phi_1)^2\, n_x\, dS + \frac{1}{4}[\,\phi_1^2(-a,0) - \phi_1^2(a,0)\,], \tag{1}$$

in the usual notation, where $\overline{S}_B$ is the mean wetted surface of the body, $(-a,0)$ and $(a,0)$ are the intersection points of the body and the mean free surface and $K = \omega^2/g$.

The functions $\psi = \partial\phi_1/\partial x$ and $\chi = -\partial\phi_1/\partial z$ both satisfy Laplace's equation in the fluid and moreover, χ is the harmonic conjugate of ψ. Green's theorem is applied to the two harmonic functions ϕ_1 and ψ yielding

$$\int_C \left[\phi_1 \frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi_1}{\partial n}\right] dS = 0 \tag{2}$$

where C is the contour consisting of the mean free surface, the mean wetted surface of the body and closing lines as $x \to \pm\infty$ and $z \to -\infty$. Under the assumption that the linear free surface condition may be differentiated tangentially to the boundary, ψ satisfies

$$K\psi - \frac{\partial \psi}{\partial z} = 0 \qquad \text{on} \qquad z = 0\,,\ |x| > a\,. \tag{3}$$

Thus, contributions to (2) arise only from the body surface and the closing lines at infinity. (It may be shown that whilst the derivatives of ψ may be singular near the intersection points of the body and the mean free surface, the singularities are not sufficiently strong to contribute to the integral in (2).) After some manipulation (2) becomes

$$\int_{\overline{S}_B} \phi_1 \frac{\partial \psi}{\partial n} + 2KR = 0\,, \tag{4}$$

where R is the first-order reflection coefficient. From the Cauchy - Riemann equations,

$$\frac{\partial \psi}{\partial n} = -\frac{\partial \chi}{\partial s} \qquad \text{on} \quad S_B\,, \tag{5}$$

where $\partial/\partial s$ is the anticlockwise, tangential derivative on the body surface. Substitution of (5) into (4) and integration by parts gives

$$\frac{1}{K}\int_{\overline{S}_B} \frac{\partial \phi_1}{\partial z}\frac{\partial \phi_1}{\partial s}\, dS = 2R + \phi_1^2(a,0) - \phi_1(-a,0)\,. \tag{6}$$

By expressing $\partial\phi_1/\partial n$ and $\partial\phi_1/\partial s$ on S_B in terms of their x and z derivatives and using the body boundary condition $\partial\phi_1/\partial n = 0$ on S_B, it may be shown that

$$\frac{1}{K}\int_{\overline{S}_B} \frac{\partial \phi_1}{\partial z}\frac{\partial \phi_1}{\partial s}\, dS = -\frac{1}{K}\int_{\overline{S}_B} (\nabla\phi_1)^2 n_x\, dS\,. \tag{7}$$

Thus, (6) and (7) may be substituted into (1) to give

$$\frac{\epsilon^2 f_{2x}^{(1)}}{\rho g A^2} = -\frac{1}{2}[\,R + \phi_1^2(a,0) - \phi_1^2(-a,0)\,]\,, \tag{8}$$

if the body intersects the free surface and

$$\frac{\epsilon^2 f_{2x}^{(1)}}{\rho g A^2} = -\frac{R}{2}\,, \tag{9}$$

if the body is submerged.

A similar analysis in three dimensions yields far-field formulae for the horizontal components of $f_2^{(1)}$ in fluid of infinte depth, namely,

$$\frac{\epsilon^2 f_{2x}^{(1)}}{\rho g A^2 a} = \frac{1}{2a}\int_\Gamma \phi_1^2 n_x\, d\Gamma - \frac{H(\pi)}{Ka}\,, \tag{10}$$

and

$$\frac{\epsilon^2 f_{2y}^{(1)}}{\rho g A^2 a} = \frac{1}{2a} \int_\Gamma \phi_1^2 n_y \, d\Gamma \,, \tag{11}$$

if the body intersects the free surface at right angles and

$$\frac{\epsilon^2 f_{2x}^{(1)}}{\rho g A^2 a} = -\frac{H(\pi)}{Ka} \,, \tag{12}$$

and

$$\frac{\epsilon^2 f_{2y}^{(1)}}{\rho g A^2 a} = 0 \,, \tag{13}$$

if the body is completely submerged. Here, $H(\theta)$ is the Kochin function, Γ is the mean waterline and the incident wave is travelling in the positive x direction. (The formulae in this section may also be derived using Tuck's theorem, see Ogilvie & Tuck (1969).)

Results and discussion

Equations (8) - (13) are expressions for the horizontal components of the oscillatory second-order force due to the first-order potential which obviate the need to evaluate $\nabla\phi_1$ on the body surface. Numerically this is more efficient, but the form of the expressions also allow deductions to be made more easily about the force on certain special bodies.

Dean (1948) proved that there is no reflection from a submerged, circular cylinder at any frequency. From (9), an immediate consequence is that the first-order potential does not contribute to the second-order oscillatory force. This result was observed by Wu & Eatock Taylor (1989) who integrated $(\nabla\phi_1)^2 n_x$ over the surface of the cylinder directly. Numerical calculations of the force on a semi-circular cylinder in the free surface were made from (8) using a mulitipole potential formulation. The predictions are in good agreement with the results obtained by Wu & Eatock Taylor (1989), by direct integration of the second-order pressure over the body. It was observed numerically that

$$\phi_1^2(a,0) - \phi_1^2(-a,0) = -4R \tag{14}$$

at all frequencies and so (8) may be rewritten as

$$\frac{\epsilon^2 f_{2x}^{(1)}}{\rho g A^2 a} = \frac{3R}{2} \tag{15}$$

for this body. However, equation (14) does not hold for all bodies, as it fails for the vertical barrier. (This may be easily demonstrated by considering the explicit representation of ϕ_1 for the vertical barrier, derived by Ursell (1947).)

The three dimesional formulae, (10) and (11), were verified for the vertical circular cylinder by comparison with the results of Kim & Yue (1989). (The formulae were modified slightly to take into account finite rather than infinite depth.) It is interesting to observe from equation (13) that the first-order potential does not contribute to the double frequency force on a submerged body, in the direction perpendicular to the incident wave advance, irrespective of any symmetry in the body shape.

Conclusion

Formulae have been derived which relate the oscillatory second-order force due to the first-order potential to the far-field amplitude of the linear, diffracted wave. The resulting expressions eliminate the need to evaluate the gradient of the first-order potential over the surface of the body and enable simple observations to be made about the force on certain special bodies. A straightforward modification of the theory whereby Green's theorem is applied to $\overline{\phi}_1$, the complex conjugate of ϕ_1 and $\partial\phi_1/\partial x$ or $\partial\phi_1/\partial y$ produces the far-field formulae for the drift force derived by Maruo (1960). Furthermore, the expression for the steady vertical moment on a three dimensional body, derived by Newman (1967), may be obtained by applying Green's theorem to $\overline{\phi}_1$ and $x\,\partial\phi_1/\partial y - y\,\partial\phi_1/\partial x$.

References

Dean, W. R. 1948 On the reflexion of surface waves by a submerged, circular cylinder. *Proc. Camb. Phil. Soc.*, Vol. 44, pp 483 - 491.

Kim, M. H. & Yue, D. K. P. 1989 The complete second-order diffraction solution for an axisymmetric body : Part I Monochromatic waves. *J. Fluid Mechanics*, Vol. 200, pp 235 - 264.

Maruo, H. 1960 The drift of a body floating in waves. *J. Ship Research*, Vol. 4, pp 1 - 10.

Mei, C. C. 1983 *The Applied Dynamics of Ocean Surface Waves* Wiley - Interscience.

Newman, J. N. 1967 The drift force and moment on ships in waves. *J. Ship Research*, Vol. 11, pp 51 - 59.

Ogilvie, T. F. & Tuck, E. O. 1969 *University of Michigan, Dept. of Naval Architecture and Marine Engineering Report No, 013*

Ursell, F. 1947 The effect of a fixed, vertical barrier on surface waves in deep water. *Proc. Camb. Phil. Soc.*, Vol. 43, pp 374 - 382.

Wu, G. X. & Eatock Taylor, R. 1989 Second order diffraction forces on horizontal cylinders. *J. Hydrodynamics*, Vol. 2, pp 55 - 65.

DISCUSSION

MOLIN: What you have done applies to a fixed body. With a moving body other terms arise in the expression of the second-order force. Have you considered how to tackle them as well?
M. McIVER: I have not considered a moving body although it would be interesting to do so. I anticipate that it would be possible to obtain alternative expressions for that component of the force arising from the quadratic term in Bernoulli's equation, using the method outlined here.

TUCK:
(a): Is the force on the horizontal submerged plate a "leading-edge suction" force equivalent to that in aerodynamics?
(b) I would surmise that it might be easier to derive directly the result with 3/2 R rather than the intermediate result with -4R. That is, I might have expected that the $\phi^2(a) - \phi^2(-a)$ terms could somehow have been absorbed in an alternative (perhaps more physically interpretable) expression.
M. McIVER:
a) I think that that must be correct, yes.
b) You may be right. However, I am unsure whether the 3R/2 result just applies to the semi-circular cylinder or whether it is true for a more general class of bodies. I will look into this further.

RAINEY: There is a very nice link with my slender-body results here, because your zero-crosswave 2nd harmonic result follows immediately from eqn 8.1 in my 1989 JFM paper, ***provided there is no 2nd order potential in the incident wave*** (thereby ensuring that is in my eqn 8.2 is purely 1st order - $\Delta e/\Delta x$ is obviously zero cross-wave). The explanation is of course that for your fixed-body case there is no 2nd order potential generated at the body surface and the effect of the 2nd order potential generated at the free surface vanishes for any slender-body case (as I argue in my paper at this workshop).
So there is no 2nd order potential contribution at all, as you require.
M. McIVER: That is interesting.

EVANS: A further special case is the fixed totally submerged matrice plate problem which I solved in 1970 (*J Fluid Mech.*). I believe I considered both the mean second order horizontal as well as vertical force. Have you checked whether it equals -1/2 R in this case?
M. McIVER: I haven't checked the result for this specific body, but I believe my result holds for general bodies.
My result is concerned with the double frequency component of the force not the mean force but Maruo's result holds for mean forces.

LIU: For a submerged horizontal circular cylinder the mean horizontal drift force F_x due to $\nabla\phi_1^{(1)}.\nabla\phi_1^{(1)}$ vanishes. The next possible contribution to F_x is from $\nabla\phi_2^{(2)}.\nabla$

$\phi_2^{(2)}$. Is it possible to apply the procedure you described for $\nabla\phi_1^{(1)}.\nabla\phi_1^{(1)}$ to obtain a similar far-field formulation for F_x due to $\nabla\phi_2^{(2)}.\nabla\phi_2^{(2)}$ or do you expect any difficulty in doing this.

M. McIVER: You could certainly apply Green's theorem to $\phi_2^{(2)}$ and $\partial\phi_2^{(2)}/\partial x$ to give an alternative expression for an integral of $\nabla\phi_2^{(2)}.\nabla\phi_2^{(2)}$ over the body. However because of the inhomogeneous free surface boundary condition at second order, you would get integrals over the free surface coming in to the expression. Thus the problem would not be simplified.