THE FOURIER INTEGRATION IN THE FOURIER-KOCHIN-GALERKIN APPROACH

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The Fourier-Kochin-Galerkin Approach

The Fourier-Kochin approach and the Galerkin solution-procedure using patches and panels presented in [1,2] and [3], respectively, and summarized in [4] offer several desirable features. An important advantage of the FKG approach is that the basic mathematical and numerical difficulties associated with the numerical evaluation of the singularities of the Green function $G(\vec{x}, \vec{\xi})$ and its gradient, and their subsequent panel-integration, are avoided as a result of two preliminary surface integrations (with respect to the points $\vec{x}$ and $\vec{\xi}$). Furthermore, a Galerkin solution-procedure utilizing patches, which may be regarded as 'higher-order' curved panels, together with flat triangular panels offers the advantage of decoupling the representation of the variation of the potential (or of equivalent source or dipole distributions) and the integration over the surface of the body; more precisely, the variation of the potential is defined by means of a Galerkin representation (employing Chebyshev polynomials) within patches, and the surface-integrations over the patches are performed by subdividing each patch into a number (> 100) of flat triangular panels. This approach makes it possible to discretize the body surface into the very large number (> 10,000) of panels required for accurary. Another appealing feature of the FKG approach is its generality: whereas traditional panel methods require considerable preliminary mathematical and numerical investments for devising reliable and efficient methods to evaluate the Green functions corresponding to specific boundary conditions, the FKG approach is directly applicable to a wide class of boundary-value problems governing dispersive waves. Indeed, the FKG approach relies wholly upon the dispersion relation characterizing the dispersive waves, as is shown in the analysis presented further on. The FKG method, however, involves one crucial nontrivial task. This task consists in numerically evaluating singular double Fourier integrals. A method for performing the Fourier integration in the FKG approach is presented in [3]. Significant modifications and improvements to the method, devised as a result of numerical experimentation, are presented here.

The Generic Fourier Integral

It is shown in [3] and [4] that the FKG approach involves Fourier integrals, R say, of the form

$$R = \lim_{\epsilon \to +0} \int_0^\infty \rho \, d\rho \int_{-\pi}^{\pi} d\theta \, N(\theta, \rho) \, / \, D_\epsilon(\theta, \rho) \,, \tag{1}$$

where ρ and θ are the polar Fourier variables corresponding to the Cartesian Fourier variables $\alpha \equiv \rho \cos\theta$ and $\beta \equiv \rho \sin\theta$, the numerator $N(\theta, \rho)$ is given by the product of two spectrum functions, which are continuous everywhere and vanish as $\rho \to \infty$, and D_ϵ is the dispersion relation involving the artificial exponential time-growth factor ϵ, in the manner shown in [5] and adopted in [1]-[4]. The dispersion relation $D_\epsilon(\theta, \rho)$ can be shown [5] to be of the general form

$$D_\epsilon(\theta, \rho) \sim D(\theta, \rho) - i\epsilon \, D'(\theta, \rho) \quad as \; \epsilon \to 0 \tag{2}$$

with $D \equiv [D_\epsilon]_{\epsilon=0}$ and $D' \equiv [i \, \partial D_\epsilon / \partial \epsilon]_{\epsilon=0}$; the functions $D(\theta, \rho)$ and $D'(\theta, \rho)$ are real. The function D represents the actual dispersion relation, which corresponds to the limit $\epsilon = 0$ of the complex dispersion relation D_ϵ. The Fourier integral (1) is considerably simpler than the corresponding Fourier integral in the expression for the Green function $G(\vec{x}, \vec{\xi})$, for which the spectrum function N is equal to $\exp[\rho(z+\zeta) + i\rho\{(x-\xi)\cos\theta + (y-\eta)\sin\theta\}]$, because N vanishes in the limit $\rho \to \infty$ whereas the exponential function in the expression for the Green function is equal to 1 at the singularity $(z = 0 = \zeta, x = \xi, y = \eta)$. However, the spectrum function $N(\theta, \rho)$ in the Fourier integral (1) is defined numerically (via an integration over the body surface) and is more complex than the exponential function in the expression for the Green function; as a result, analytical techniques, notably contour integration, which may be used for evaluating the Green function cannot be applied to evaluate the Fourier integral (1), which must then necessarily be evaluated numerically.

The Limit $\epsilon \to +0$ And The Far-Field Waves

The Fourier integral (1) may be expressed in the form

$$R = R_0 + \lim_{\epsilon \to +0} R_\epsilon \,, \tag{3}$$

where R_0 and R_ϵ are the integrals defined as

$$R_0 = \int_0^\infty \rho\, d\rho \int_{-\pi}^{\pi} d\theta\, N/D\,, \qquad R_\epsilon = \int_0^\infty \rho\, d\rho \int_{-\pi}^{\pi} d\theta\, N\, A_\epsilon \quad with \quad A_\epsilon \equiv 1/D_\epsilon - 1/D\,. \tag{4}$$

The integral R_ϵ is now considered. We have $A_\epsilon = (D-D_\epsilon)/(DD_\epsilon) = i\epsilon D'/[D\,(D-i\epsilon D')]$ by virtue of (2). It may then be seen that $A_\epsilon \to 0$ as $\epsilon \to 0$ if $D \neq 0$. The only contribution to the integral R_ϵ therefore stems from the curves in the Fourier plane where we have $D=0$. The dispersion equation $D=0$ generally defines one or several distinct curves in the Fourier plane, which are referred to as dispersion curves hereafter. The dispersion curves corresponding to the waves generated by a ship advancing in regular waves and to several other dispersive waves are depicted in [1]-[3] and [5], respectively. The Fourier integral R_ϵ thus may be expressed in the form

$$R_\epsilon = \sum_{D=0} \int_{D=0} N(s)\, I_\epsilon(s)\, J(s)\, ds \qquad with \quad I_\epsilon(s) = \int_{-\delta}^{\delta} A_\epsilon(D,s)\, dD\,, \tag{5}$$

where $\sum_{D=0}$ means summation over the various dispersion curves $D=0$, s is the arc length along the corresponding dispersion curve, $N(s)$ represents the value of the spectrum function $N(D,s)$ along the dispersion curve, δ is an arbitrary small real positive number, and $J(s)$ is the value at the dispersion curve of the Jacobian $J(D,s)$ associated with the coordinate transformation $(\rho,\theta) \to (D,s)$. The Jacobian J is defined by the relation $\rho\, d\rho\, d\theta \equiv J\, dD\, ds \equiv J\,(\partial D/\partial n)\, dn\, ds$, where n represents the distance along the normal to the dispersion curve $D=0$. We also have $\rho\, d\rho\, d\theta \equiv dn\, ds$. It follows that the Jacobian J is given by $J = 1/(\partial D/\partial n)$. The vector ∇D is normal to the dispersion curve $D=0$, and we thus have $\partial D/\partial n \equiv \nabla D \bullet \vec{n} = \nabla D \bullet \nabla D/\|\nabla D\| = \|\nabla D\|$. This yields

$$J = 1/\|\nabla D\| \equiv 1/\sqrt{(\partial D/\partial \alpha)^2 + (\partial D/\partial \beta)^2} \equiv 1/\sqrt{(\partial D/\partial \rho)^2 + (\partial D/\partial \theta)^2/\rho^2}\,. \tag{6}$$

The integral $I_\epsilon(s)$ in (5) is now considered. We have $A_\epsilon = i\epsilon D'/[D\,(D-i\epsilon D')]$ as was already noted. We then have $A_\epsilon = i\epsilon D'(1+i\epsilon D'/D)/[D^2+\epsilon^2(D')^2]$. By substituting this expression for A_ϵ into (5), performing the change of variable $D = \epsilon\lambda$ (with $\epsilon > 0$), and taking the limit $\epsilon \to 0$ we may obtain

$$\lim_{\epsilon \to +0} I_\epsilon(s) = iD' \int_{-\infty}^{\infty} d\lambda\, (1+iD'/\lambda)\, /\, [\lambda^2 + (D')^2]\,.$$

The imaginary part of the integrand is an odd function and the corresponding integral thus is null. The integral can then be shown to be equal to $i\pi\, \mathrm{sign} D'$. By using this result in (5) we may finally obtain

$$\lim_{\epsilon \to +0} R_\epsilon = i\pi \sum_{D=0} \int_{D=0} sign D'(s)\, N(s)\, J(s)\, ds\,. \tag{7}$$

The Fourier integral R defined by (1) thus is expressed in (3) as the sum of the double integral R_0 defined by (4), which corresponds to the limit $\epsilon=0$, and the single integral along the dispersion curves defined by (7). The latter integral can be shown to represent the system of far-field waves, while the integral R_0 corresponds to a nonoscillatory near-field flow disturbance. The integrand of the double integral R_0 defined by (4) is singular along the dispersion curves $D=0$, as was already noted. This singular double integral is now considered.

Analytical Integration Of The Singularity

Let us consider an even function $E(\Delta)$ that is equal to 1 for $\Delta = 0$, nearly equal to 1 for small values of Δ, and negligibly small outside the range $-1 \leq \Delta \leq 1$. For instance, the function $E(\Delta)$ may be chosen as $E(\Delta) = \exp[-3\Delta^2(2+7\Delta^6)/4]$ as is recommended in [3]. Furthermore, let $\Delta \equiv D/\delta$. The function $E(D/\delta)$ thus is equal to 1 along every dispersion curve $D=0$, nearly equal to 1 in their immediate vicinities, and negligibly small outside the strips defined by $-\delta \leq D \leq \delta$. The width of these strips, referred to as dispersion strips hereafter, is controlled by the parameter δ. It is desirable that the width, equal to $2\,dn$, of the dispersion strips be constant. This condition can be satisfied by choosing the parameter δ as the function $\delta = \kappa\,\|\nabla D\| = \kappa\,/\,J$, where κ is a constant equal to half the width of the dispersion strips, and J is the Jacobian defined by (6). The relation $\delta = \kappa \|\nabla D\|$ follows from the identities $\delta \equiv dD \equiv (\partial D/\partial n)\, dn \equiv \|\nabla D\|\, dn$ and $dn = \kappa$.

The singular double Fourier integral R_0 defined by (4) is expressed in the form

$$R_0 = R_2 + R_S\,, \tag{8}$$

where the integrals R_2 and R_S are defined as

$$R_2 = \int_0^\infty \rho\, d\rho \int_{-\pi}^{\pi} d\theta\, N\,[1-E(DJ/\kappa)]\,/\,D\,, \qquad R_S = \int_0^\infty \rho\, d\rho \int_{-\pi}^{\pi} d\theta\, N\, E(DJ/\kappa)\,/\,D\,; \tag{9}$$

The integrand of the integral R_2 is continuous everywhere. In particular, it vanishes along the dispersion curves $D=0$. Numerical evaluation of the integral R_2 thus presents no essential difficulty.

The singular integral R_S is now considered. The exponential function $E(DJ/\kappa)$ is negligibly small outside the narrow strips $-\kappa/J \le D \le \kappa/J$, as was already noted. The integral R_S may then be expressed in the form

$$R_S = \sum_{D=0} \int_{D=0} I(s)\, ds \qquad with \quad I(s) = \int_{-\infty}^{\infty} dD\, J(D,s)\, N(D,s)\, E\{D\, J(D,s)\,/\kappa\}\,/\,D\,, \tag{10}$$

where $J(D,s)$ is the previously-defined Jacobian associated with the coordinate transformation $(\rho,\theta) \to (D,s)$, and the limits of integration for the integral $I(s)$ are taken as ∞ since the function $E(DJ/\kappa)$ is negligibly small for $|D| > \kappa/J$. The change of variable $\lambda = D\, J(D,s)$ yields

$$I(s) = \int_{-\infty}^{\infty} d\lambda\, \widetilde{N}(\lambda,s)\, E(\lambda/\kappa)\,/\lambda \qquad with \quad \widetilde{N} \equiv NJ/[\,1+D\,(\partial J/\partial D)\,/J\,]\,. \tag{11}$$

The function $\widetilde{N}(\lambda,s)$ can be expanded in a Taylor series about the dispersion curve $\lambda=0$, as follows: $\widetilde{N}(\lambda,s) = \widetilde{N}(s) + \lambda\, \widetilde{N}'(s) + \lambda^2\, \widetilde{N}''(s)\,/2 + \lambda^3\, \widetilde{N}'''(s)\,/6 + \cdots$, where $\widetilde{N}^{(k)}(s) \equiv [\partial^k \widetilde{N}(\lambda,s)/\partial\lambda^k]_{\lambda=0}$. The part of the integrand of the integral $I(s)$ in (10) that corresponds to the even terms $\widetilde{N}(s)+\lambda^2\, \widetilde{N}''(s)\,/2+\cdots$ are odd functions of λ. We then have $I(s) = C\,\kappa\, \widetilde{N}'(s) + O[C_3\, \kappa^3\, \widetilde{N}'''(s)/6]$, where the constants C and C_3 are defined as $C = \int_{-\infty}^{\infty} E(\Delta)\, d\Delta \simeq 1.16724$ and $C_3 = \int_{-\infty}^{\infty} \Delta^2\, E(\Delta)\, d\Delta \simeq 0.18263$; we thus have $C_3/6 \simeq 0.03$.

If the arbitrary parameter κ, which defines the width of the dispersion strip, is sufficiently small, we have

$$I(s) \simeq C\,\kappa\,[\partial \widetilde{N}(\lambda,s)/\partial\lambda\,]_{\lambda=0} \qquad with \quad C = \int_{-\infty}^{\infty} E(\Delta)\, d\Delta \simeq 1.16724\,. \tag{12}$$

We have $\partial\widetilde{N}/\partial\lambda = (\partial\widetilde{N}/\partial D)\,/\,(d\lambda/dD) = (\partial\widetilde{N}/\partial D)\,/\,(J+D\,\partial J/\partial D)$; it follows that we have $[\partial\widetilde{N}/\partial\lambda\,]_{\lambda=0} = [\partial\widetilde{N}/\partial D\,]_{D=0}\,/J(s)$, where $J(s)$ represents the value of the Jacobian $J(D,s)$ at the dispersion curve $D=0$. Furthermore, (11) yields $[\partial\widetilde{N}/\partial D\,]_{D=0} = J(s)\,[\partial N/\partial D\,]_{D=0}$. We thus have $[\partial\widetilde{N}/\partial\lambda\,]_{\lambda=0} = [\partial N/\partial D\,]_{D=0}$. We have $\partial N/\partial D = (\partial N/\partial n)\,/\,(\partial D/\partial n) = J\,\partial N/\partial n$ since $J = 1/(\partial D/\partial n)$ as was previously shown. It follows that $[\partial\widetilde{N}/\partial\lambda\,]_{\lambda=0} = J(s)\,[\partial N/\partial n\,]_{D=0}$. By substituting this expression into (12) and using (10) we may obtain

$$R_S \simeq C\,\kappa \sum_{D=0} \int_{D=0} [\partial N/\partial n\,]_{D=0}\, J(s)\, ds\,. \tag{13}$$

The singular double integral R_S in (9) thus is expressed as an integral along the dispersion curves $D=0$, and the singular double integral R_0 in (3), (4) and (8) is expressed as the sum of the regular double integral R_2 defined by (9) and the integral along the dispersion curves R_S defined by (13).

Summary Of Results

The integral R_S in (13) is of the same form as the integral $\lim_{\epsilon\to+0} R_\epsilon$ in (7) but involves the derivative $\partial N/\partial n$ of the spectrum function N in the direction normal to the dispersion curves $D=0$ instead of the function N. The integral R_S in (13) and the integral $\lim_{\epsilon\to+0} R_\epsilon$ in (7) may be grouped. Specifically, (3) and (8) yield

$$R = R_1 + R_2\,, \tag{14}$$

where we have $R_1 \equiv \lim_{\epsilon\to+0} R_\epsilon + R_S$. It may then be seen from (7), (13) and (9) that the integrals R_1 and R_2 in (14) are defined as

$$R_1 \simeq \sum_{D=0} \int_{D=0} ds\,(\,i\pi\, signD'\, N + C\kappa\, \partial N/\partial n\,)\,/\,\|\nabla D\|\,, \qquad R_2 = \int_0^\infty \rho\, d\rho \int_{-\pi}^{\pi} d\theta\, N\,[\,1-E(\kappa' D/\,\|\nabla D\|\,)\,]\,/\,D, \tag{15}$$

where (6) was used. The parameter $\kappa' \equiv 1/\kappa$ controls the width of the dispersion strips. The constant C is defined as $C = \int_{-\infty}^{\infty} E(\Delta)\, d\Delta$. If the function $E(\Delta)$ is chosen as $E(\Delta) = \exp[-3\Delta^2(2 + 7\Delta^6)/4]$, as is recommended, we have $C \simeq 1.16724$. The term $\partial N/\partial n$ is given by $\nabla N \bullet \vec{n} = \nabla N \bullet \nabla D / \|\nabla D\|$. The expression (15) becomes exact in the limit $\kappa \to 0$. However, the function $E(\kappa' D/\|\nabla D\|)$ in the integrand of the double integral R_2 in (15) vanishes in this limit and the integral R_2 thus becomes singular at the dispersion curves $D=0$. The foregoing expression is valid for dispersive waves characterized by a dispersion relation of the general form given by (2).

Application To The Problem Of Ship Motions

In the particular case of a ship advancing in regular waves the dispersion relation D_ϵ in (2) takes the form $D_\epsilon \equiv (f + i\epsilon - F\rho\cos\theta)^2 - \rho$, where $f \equiv \omega\sqrt{L/g}$ and $F \equiv U/\sqrt{gL}$ are the nondimensional frequency and Froude number, with $\omega =$ wave frequency, $L =$ ship length, $U =$ ship speed, and $g =$ acceleration of gravity. We thus have $D \equiv (f - F\rho\cos\theta)^2 - \rho$ and $D' \equiv 2(F\rho\cos\theta - f)$. It follows that we have $sign D' = sign(F\rho\cos\theta - f)$. Furthermore, we have $D_\rho \equiv \partial D/\partial\rho = -1 - 2A\cos\theta$ and $D_\theta \equiv \partial D/\partial\theta = 2A\rho\sin\theta$, where A is defined as $A \equiv \tau - F^2\rho\cos\theta$ with $\tau \equiv fF \equiv U\omega/g$. We thus have $\|\nabla D\| = \sqrt{1 + 4(\tau - F^2\alpha)\cos\theta + 4(\tau - F^2\alpha)^2}$ with $\alpha \equiv \rho\cos\theta$. In the special case $F=0$, corresponding to diffraction-radiation without forward speed, we have $\|\nabla D\| = 1$. At the dispersion curve $D=0$ we have $\rho = (F\alpha - f)^2$, as follows from the dispersion relation. We may then obtain $\|\nabla D\| = \sqrt{1 + 4(\tau - F^2\alpha)\cos\theta + 4F^2\rho}$, which becomes $\|\nabla D\| = \sqrt{4F^2\rho - 3} = \sqrt{1 + 4\tan^2\theta}$ in the special case $f=0$, that is for steady flows. It may also be seen that we have $\nabla N \bullet \nabla D \equiv N_\rho D_\rho + N_\theta D_\theta/\rho^2 \equiv -N_\rho - 2A(\cos\theta\, N_\rho - \sin\theta\, N_\theta/\rho)$. Furthermore, we have $\cos\theta\, N_\rho - \sin\theta\, N_\theta/\rho \equiv N_\alpha$ where $N_\alpha \equiv \partial N/\partial\alpha$. We thus have $\nabla N \bullet \nabla D = -[N_\rho + 2(\tau - F^2\alpha)N_\alpha]$ and $\partial N/\partial n = -[N_\rho + 2(\tau - F^2\alpha)N_\alpha]/\sqrt{1 + 4(\tau - F^2\alpha)\cos\theta + 4F^2\rho}$.

The dispersion relation $D(\theta, \rho)$ is an even function of θ; the Fourier integration in (15) may then be restricted to the upper half plane $0 \le \theta \le \pi$. Furthermore, the upper half of the Fourier plane is subdivided into the sectors $[\beta \le \theta \le \pi - \gamma]$ and $[0 \le \theta \le \beta] \cup [\pi - \gamma \le \theta \le \pi]$. Within these two sectors, the dispersion curves $D=0$ are defined by the equations $\cos\theta = (f/\sqrt{\rho} \pm 1)/(F\sqrt{\rho})$ and $4F^2\rho = [(1 \pm \sqrt{1 + 4\tau\cos\theta})/\cos\theta]^2$, respectively. The corresponding values of the functions $sign D'$ and $\partial N/\partial n$ within these sectors are given by $sign D' = \pm 1$ and $\partial N/\partial n = -(N_\rho \mp 2F\sqrt{\rho}\, N_\alpha)/\sqrt{4(F^2\rho \mp f/\sqrt{\rho}) - 3}$, and $sign D' = sign[\cos\theta\,(1 \pm \sqrt{1 + 4\tau\cos\theta})]$ and $\partial N/\partial n = -[N_\rho - N_\alpha(1 \pm \sqrt{1 + 4\tau\cos\theta})/\cos\theta]/\sqrt{1 + 4\tau\cos\theta + (1 \pm \sqrt{1 + 4\tau\cos\theta})\tan^2\theta}$, respectively. The integral R_1 along the dispersion curves in the two sectors is expressed as an integral with respect to ρ and θ, respectively, and the integrations with respect to θ and ρ in the double integral R_2 are performed in the orders $\int d\rho \int d\theta$ and $\int d\theta \int d\rho$, respectively. This partition of the Fourier plane into $\int d\rho \int d\theta$ and $\int d\theta \int d\rho$ sectors is dictated by the shape of the dispersion curves, that is by the direction in which energy is radiated via the surface waves. We thus have

$$R = \int_0^\infty [\sum_{D=0} \sigma(\rho)\, I_1(\rho) + \rho\, I_2(\rho)]\, d\rho + \int_{[0,\beta]\cup[\pi-\gamma,\pi]} [\sum_{D=0} \sigma(\theta)\, I_1(\theta) + I_2(\theta)]\, d\theta,$$

$$with \qquad I_1 = i\pi\, sign D' \widehat{N} + C\kappa\, \partial\widehat{N}/\partial n, \qquad \sigma(\rho) \equiv (ds/d\rho)/\|\nabla D\| \qquad and \qquad \sigma(\theta) \equiv (ds/d\theta)/\|\nabla D\|;$$

furthermore, the terms $I_2(\rho)$ and $I_2(\theta)$ are defined as

$$I_2(\rho) = \int_\beta^{\pi-\gamma} d\theta\, \widehat{N}(\theta;\rho)\, [1 - E(\kappa' D/\|\nabla D\|)]\, /\, D(\theta;\rho), \quad I_2(\theta) = \int_0^\infty d\rho\, \rho\, \widehat{N}(\rho;\theta)\, [1 - E(\kappa' D/\|\nabla D\|)]\, /\, D(\rho;\theta),$$

and we have $\widehat{N}(\theta, \rho) \equiv N(\theta, \rho) + N(-\theta, \rho)$. The function $\sigma(\rho)$ is given by $2\sigma(\rho) = 1/\sqrt{F^2\rho - (1 \pm f/\sqrt{\rho})^2}$, and the function $\sigma(\theta)$ by $\sigma(\theta) = \rho(\theta)/\sqrt{1 + 4\tau\cos\theta}$ with $4F^2\rho(\theta) = [(1 \pm \sqrt{1 + 4\tau\cos\theta})/\cos\theta]^2$.

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