INSIGHTS INTO 2ND-ORDER DIFFRACTION COMPUTATIONS, FROM SLENDER-BODY THEORY

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Abstract

Second-order Stokes'-expansion computations of survival-wave loads on tension-leg oil rigs have recently been compared with a new slender-body theory. To illuminate the results obtained, we consider here the 2-D problem of a thin horizontal circular cylinder fully immersed parallel to the crests of deep-water regular waves, and *oscillating horizontally*. The slender-body theory shows immediately that the second-order loads are the steady vertical one which alone would be felt on a fixed cylinder, plus steady and oscillatory loads, both horizontally and vertically, tracable to the position-dependence of the first-order load. Precisely the same answer is obtained below by Stokes' expansion, for which a closed-form solution is obtained using complex potentials. This time, the loads are tracable partly to the dynamic pressure, partly to the first-order pressure-gradient, and partly to the second-order velocity potential. In all three cases they appear as the "difference of two large numbers", which is consistent with the well-known requirement for very fine discretisations with second-order Stokes'-expansion computations.

1. Slender-body Theory Result

The new slender-body theory in question here is that described in Rainey (1989), according to equations (8.1)-(8.3) of which the load (per unit cylinder length, including the fluid reaction to cylinder acceleration) is:

$$-\boldsymbol{M}\dot{\boldsymbol{u}} + \boldsymbol{M}'\left(\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{V}\boldsymbol{u}\right) + \boldsymbol{V}\boldsymbol{M}'(\boldsymbol{v}-\boldsymbol{u}) \tag{1}$$

Here $\boldsymbol{M}$ and $\boldsymbol{M}'$ are the tensors of added mass and added-mass-plus-displaced-mass (both per unit length), $\boldsymbol{v}$ and $\boldsymbol{V}$ are the velocity vector and velocity gradient tensor in the undisturbed incident wave, and $\boldsymbol{u}$ is the velocity vector of the cylinder (bold type denoting vectors/tensors). For our case of a circular cylinder $\boldsymbol{M}$ is ρc, and $\boldsymbol{M}'$ is $2\rho c$, where ρ is the water density and c the cylinder cross-sectional area, so this reduces to:

$$-\rho c\dot{\boldsymbol{u}} + 2\rho c\left(\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{V}\boldsymbol{v}\right) = -\rho c\dot{\boldsymbol{u}} + 2\rho c\boldsymbol{a} \tag{2}$$

where $\boldsymbol{a}$ is the water particle acceleration in the undisturbed incident wave.

For comparison with Stokes' expansion, we now extract from this expression the terms of second order in waveheight. We will assume henceforth for simplicity that the oscillation is exactly sinusoidal, so there are no such terms from the first (inertial reaction) part. They therefore come from the second part. At a fixed location in regular deep-water waves where the horizontal water velocity is $v\cos\omega t$, say, the second-order constituent of $\boldsymbol{a}$ is well-known to be a steady vertical acceleration of magnitude kv^2, matching the steady vertical gradient of the dynamic pressure. This gives a steady upwards force of $2\rho ckv^2$, which is the only second-order load if the cylinder is fixed.

However, since (2) above applies at the instantaneous displaced position of the cylinder (rather than its mean position as in Stokes' expansion), the motion of the cylinder produces a slight oscillation in the phase of the first-order load, as the cylinder moves upwave and downwave. This has been recognised for some time (see Rainey, 1978, discussion, & Rainey, 1980, where it is shown experimentally that it can actually lead to a dynamic instability!) as another source of second-order load. Adopting the standard complex-load notation (Batchelor, 1967, p.433 - the real part is the horizontal load and the imaginary part is minus the upward-vertical load) the first-order load on the cylinder, at its instantaneous displaced position x, is, from (2):

$$2\rho cv\omega ie^{i(\omega t-kx)} = 2\rho cv\omega ie^{i\omega t}[1-ikx+...] \tag{3}$$

Here x, the horizontal water velocity $v\cos\omega t$ above, the direction of wave travel, and the horizontal load, are all taken to the right (and we can clearly omit the inertial reaction). If we allow for arbitrary amplitude and phase of the horizontal cylinder motion by writing:

$$\dot{x} = \Re[rve^{i\omega t}] = \frac{[rve^{i\omega t}+\bar{r}ve^{-i\omega t}]}{2} \tag{4}$$

where r is complex (and where r = 1 corresponds to the cylinder moving horizontally with the water particles, to first order), the total complex second-order load becomes:

$$-i\rho ckv^2[(2-\bar{r}) + re^{2i\omega t}] \tag{5}$$

including the steady vertical load $2\rho ckv^2$ found earlier. In general, evidently, the motion modifies this load, adds a steady horizontal load (a "drift force", in the offshore parlance), and adds both horizontal and vertical second-harmonic loads.

2. Stokes' Expansion: The First-Order Complex Potential

To keep the algebra manageable in a Stokes' expansion analysis of the problem, it is necessary to use complex potentials. We first write the velocity potential of the incident waves as:

$$-\frac{v}{k}e^{ky}\sin(\omega t-kx) \quad = \quad \Re\left[\frac{iv}{k}e^{i(\omega t-kz)}\right] \tag{6}$$

where position coordinates x, y, in the direction of wave travel and vertically upwards (measured from the mean posion of the cylinder), are replaced by the single complex coordinate $z = x+iy$, in the standard manner (Batchelor, 1967, Sect 6.5). To describe the flow near the cylinder, we next expand this potential as a Taylor series about $z = 0$, thus:

$$\frac{iv}{k}e^{i\omega t}\left[1-ikz+\frac{(-ikz)^2}{2}+\ldots\right] \tag{7}$$

where the z term represents the uniform flow, and the z^2 represents the simple "extensional" flow non-uniformity - no further terms being necessary for our limiting case of a thin cylinder (Lighthill, 1979; Rainey 1989 Sect.2).

The first-order diffracted potential in Stokes' expansion is that produced by our cylinder fixed at its mean position $z = 0$, which is readily given by the "circle theorem" (Batchelor, 1967, p.422) as:

$$\frac{-iv}{k}e^{-i\omega t}\left[ik\left(\frac{b^2}{z}\right)-\frac{k^2}{2}\left(\frac{b^2}{z}\right)^2\right] \tag{8}$$

where b is the radius of the cylinder. The first term is a simple dipole reaction to the uniform flow, and the second a quadrupole reaction to the incident "extensional motion". To this must be added the first-order radiated potential, which is that produced by applying the cylinder-velocity boundary-condition, but at the mean position of the cylinder. This gives via (4) another simple dipole:

$$-\frac{1}{2}\left[rve^{i\omega t}+\bar{r}ve^{-i\omega t}\right]\frac{b^2}{z} \tag{9}$$

The sum of (7),(8)&(9) is the first-order complex potential w_1. We will need below the first-order complex velocity dw_1/dz; collecting terms this is:

$$ve^{i\omega t}\left[\frac{r}{2}\frac{b^2}{z^2}+1-ikz\right] \; + \; ve^{-i\omega t}\left[\left(\frac{\bar{r}}{2}-1\right)\frac{b^2}{z^2}-ik\frac{b^4}{z^3}\right] \tag{10}$$

3. Stokes' Expansion: The Dynamic Pressure Load

Perhaps the simplest second-order load in Stokes' expansion results from integrating the dynamic pressure over the body surface; following standard complex-variable techniques (Batchelor, 1967, p.433) we can write this load (per unit cylinder length) in complex form as:

$$\frac{1}{2}i\rho\int\frac{dw_1}{dz}\left(\overline{\frac{dw_1}{dz}}\right)\overline{dz} \quad = \quad \frac{1}{2}\rho\int_0^{2\pi}\frac{dw_1}{dz}\left(\overline{\frac{dw_1}{dz}}\right)e^{-i\theta}b\,d\theta \tag{11}$$

where θ is the angle measured anticlockwise around the cylinder from the downwave (x) direction. To evaluate the integral, we can, from (10), first write down dw_1/dz on the surface of the cylinder as:

$$ve^{i\omega t}\left[\frac{r}{2}e^{-2i\theta}+1-ikbe^{i\theta}\right] \quad + \quad ve^{-i\omega t}\left[\left(\frac{\bar{r}}{2}-1\right)e^{-2i\theta}-ikbe^{-3i\theta}\right] \tag{12}$$

and then write down its complex conjugate as:

$$ve^{-i\omega t}\left[\frac{\bar{r}}{2}e^{2i\theta}+1+ikbe^{-i\theta}\right] \quad + \quad ve^{i\omega t}\left[\left(\frac{r}{2}-1\right)e^{2i\theta}+ikbe^{3i\theta}\right] \tag{13}$$

In the product of (12) and (13), only the $e^{i\theta}$ terms will survive the integration, because of the $e^{-i\theta}$ term in (11) The complex second-order load (per unit length) from the dynamic pressure is therefore:

$$\frac{1}{2}\rho kbv^2\int_0^{2\pi}\left[\left(-i+i\left(\frac{\bar{r}}{2}-1\right)\right)+i\frac{r}{2}e^{2i\omega t}\right]b\,d\theta \quad = \quad -i\rho ckv^2\left[\left(2-\frac{\bar{r}}{2}\right)-\frac{r}{2}e^{2i\omega t}\right] \tag{14}$$

In a general-purpose computer program of course, every term will be retained in the pressure integration. and all the cancellation just invoked will take place numerically. Strikingly, there are cancelling pressure terms, with an angular dependence up to $e^{3i\theta}$, which are $(kb)^{-1}$ times the non-cancelling one. For a typical tension leg oil rig in survival waves, $(kb)^{-1}$ is 10, so in carring out its numerical integration, the computer program will be up against a "signal-to-noise ratio" of 0.1.

4.Stokes' Expansion: The Pressure-Gradient Load

Another second-order load in Stokes' expansion is that produced by the motions of the cylinder through the first-order pressure gradients. In our case the motions of the cylinder are purely horizontal, so we require the horizontal gradient of first-order pressure, which is $-\rho$ times the rate-of-change of first-order horizontal velocity. Thus this second-order load can be written in complex form as:

$$\rho\int_0^{2\pi}[\frac{1}{2}(\frac{d\dot{w}}{dz}+\overline{\frac{d\dot{w}}{dz}})(\frac{rve^{i\omega t}-\bar{r}ve^{-i\omega t}}{2i\omega})e^{-i\theta}bd\theta \tag{15}$$

We therfore now seek $e^{i\theta}$ terms in the sum of (12) & (13), rather than in their product, as we did above. This time there is only one, so that our integral reduces to:

$$\frac{1}{2}\int_0^{2\pi}(-ikbv\dot{e}^{i\omega t})(\frac{rve^{i\omega t}-\bar{r}ve^{-i\omega t}}{2i\omega})bd\theta \quad = \quad -i\rho ckv^2[-\frac{\bar{r}}{2}+\frac{r}{2}e^{2i\omega t}] \tag{16}$$

where, just as above, there are cancelling pressure terms $(kb)^{-1}$ times larger, again with an angular dependence up to $e^{3i\theta}$. When (16) is added to the second-order load from dynamic pressure (14), we have the same steady load as that obtained by slender-body theory (5). This is to be expected, since the second-order potential, below, cannot generate a steady load.

5. Stokes' Expansion: The Second-Order Potential, and its Load

If we write the first-order complex velocity (10) as $\zeta - i\eta$, in other words denote the first-order horizontal and vertical velocity components as ζ and η, then we note that:

$$\frac{d^2w_1}{dz^2} = \frac{\partial\zeta}{\partial x} - i\frac{\partial\eta}{\partial x} \tag{17}$$

since we are at liberty to differentiate a complex variable by choosing dz parallel to the x-axis. The second-order error in the first-order cylinder-surface boundary condition, arising from a horizontal cylinder motion δx, can therefore be written:

$$\delta x(\frac{\partial\zeta}{\partial x}\cos\theta + \frac{\partial\eta}{\partial x}\sin\theta) = \Re[\delta x\frac{d^2w_1}{dz^2}e^{i\theta}] \tag{18}$$

with the sense being outwards from the cylinder surface. Inserting x from (4) for δx, and d^2w_1/dz^2 from (10), the second-harmonic constituent of this expression becomes:

$$\frac{v^2}{4b\omega i}[re^{2i\omega t}(-re^{-2i\theta}-ikbe^{i\theta}+(2-r)e^{2i\theta}-3ikbe^{3i\theta}) - \bar{r}e^{-2i\omega t}(-\bar{r}e^{2i\theta}+ikbe^{-i\theta}+(2-\bar{r})e^{-2i\theta}+3ikbe^{-3i\theta})] \tag{19}$$

the steady consituent being immaterial, of course, since it gives no load. There are also second-order errors in the first-order boundary condition at the free surface, which are well-known to give zero second-order incident potential, but lead to the celebrated "microseism effect" on tension leg platforms in short waves (see esp. Newman, 1990), named after the well-known analogous oceanographic phenomenon (Wehausen & Laitone, 1960, pp 665-666). In our limiting case of a thin cylinder, however, these vanish in comparison with (19), producing a load proportional to c^2 rather than c as below. See Rainey (1989 Sect.4), and, for a fixed vertical cylinder, Lighthill 1979. In our case of a fully-immersed cylinder we can see that this must be so because the free-surface boundary-condition error will clearly fall with cylinder diameter - and the resulting second-order potential already produces a load proportional to c, because it acts like another incident potential.

The relevant second-order potential is therefore just that required to cancel the outward flux (19). That flux has the same angular dependence as the outward flux produced by a sum of multipoles b^n/nz^n, so the complex second-order potential w_2 can be readily seen to be:

$$\frac{iv^2}{2\omega}[e^{2i\omega t}(r^2\frac{b^2}{2z^2}) + e^{-2i\omega t}(\bar{r}ikb\frac{b}{z}+\bar{r}(2-\bar{r})\frac{b^2}{2z^2}+3\bar{r}ikb\frac{b^3}{3z^3})] \tag{20}$$

i.e. a dipole, two quadrupoles, and an octupole. The second-order complex force produced by it is:

$$\rho \int_0^{2\pi} \frac{1}{2}(\dot{w}_2 + \overline{\dot{w}_2}) e^{-i\theta} b d\theta \quad = \quad -i\rho c k v^2 [r e^{2i\omega t}] \tag{21}$$

where the integration once again involves the cancellation of terms $(kb)^{-1}$ times larger than the non-cancelling term, with an angular dependence again up to $e^{3i\theta}$.

Very satisfactorily, we see that this third Stokes'-expansion load, in combination with the other two from Section 3 & 4, adds up as it should to the load predicted by slender-body theory in Section 1.

6. Conclusions

In a sense, there is nothing new in the above results - they merely confirm in a particular case what is already established in Rainey (1989), namely that a new development in slender-body theory allows wave loads on thin members of offshore structures to be calculated to second order in waveheight. However, the example studied does perhaps illuminate the following points:

1)Horizontal drift forces do not, in general, depend on interference effects between structural members. It has sometimes been argued (e.g. Pizer, 1990, discussion) that since they *do* depend on interference effects in the case of *fixed, vertical* cylinders, and since the slender-body theory omits them, then it must be of dubious value on, say, semisubmersible oil rigs in long waves. The above example shows that once a rig moves, its members will in general feel a horizontal drift force, which will increasingly dwarf that due to member interactions, as the members become thinner relative to the wavelength.

2)The second-order potential arising from the body-surface boundary condition, rather than that arising from the free-surface boundary condition, controls the loading on moving structural members as they become thinner relative to the wavelength. This is interesting in view of the remarkable importance of the latter potential in short waves (again see esp. Newman, 1990): once more it appears dangerous to generalise from fixed structures, on which the former potential is zero.

3)As structural members become thinner compared with a wavelength, Stokes' expansion compuations increasingly mean finding the "difference of two large numbers". In the example considered here, there are cancelling pressures $(kb)^{-1}$ (=10, typically, on a TLP in survival waves) times as large as the required non-cancelling pressures. And they have a finer spatial dependence than first-order quantities. This is consistent with the well-known requirement for much finer discretisations with second-order Stokes' expansion compuations, than with first-order ones.

4)By contrast the slender-body theory uses the same simple well-conditioned parameter (2-D added mass) at both first and second order. This is another striking illustration of the point stressed in Rainey (1989), that a *simpler* flow model can be used, for the *same* level of accuracy (i.e. second-order) when surface-pressure integration is avoided. The analogous case featured there is Taylor (1928), who required a lower-order multipole expansion when he used an energy argument, than he did to obtain the same answer by surface pressure integration.

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DISCUSSION

M. McIVER: What is the size of the wavelength compared to the wave amplitude?
RAINEY: Arbitrary — the "wavy lid" described in Rainey (1989) can be shaped like a breaking wave, for example. However, at a surface intersection the pressures across it will be large (due to the $\frac{1}{2}\rho V^2$ dynamic pressure term) unless the waveheight is small compared with the cylinder diameter. This restriction is discussed in Rainey (1989) p. 300, and applies to Stokes' expansion too, of course.

MILOH: Can you be kind enough and elaborate a bit more on why is (to your opinion) the energy approach preferable to the commonly used momentum method, which involve pressure integration, for the titled problem?
RAINEY: I believe the momentum method is harder going algebraically, because of the need to consider the reactions at far boundaries (see e.g. Newman's "Marine Hydrodynamics", fig. 7.2), which do ***not*** vanish as the boundaries are made more distant. Also, I have never seen how the rate-of-change-of-energy-with-position term ($\Delta e/\Delta x$ in my 1989 JFM paper) comes out of a momentum argument, although presumably it does somehow.

VAN DAALEN: In your 2nd sheet you showed the increase of angular momentum of the fluid due to the propagation of the cylinder. My comment on this figure is that it contains the basic concept of one of the eight conservation laws for a fluid with a free surface including a floating body. The underlying theory has been treated extensively by Benjamin and Olver (JFM 1982) for water waves only. Recently, I extended their theory to water waves interacting with freely floating bodies. Besides energy, mass and momentum, some other quantities are conserved as well for the coupled wave-body system. One of these quantities is the total angular momentum with respect to a fixed point, e.g. the center of mass of the system. The conservation laws and the proofs will be presented in my Ph.D. thesis.
RAINEY: I look forward to seeing your Ph.D. thesis. There are interesting difficulties with momentum arguments, as I indicated in reply to Prof. Miloh. To my knowledge, general momentum arguments have hitherto been restricted to first order in waveheight, see e.g. Newman's "Marine Hydrodynamics", section 7.7.

FALTINSEN: You mention the TLP application. We found that the surge damping on a TLP, for example, is only a potential flow effect ("wave drift" damping) in the smaller waves. In survival waves, it is dominated by the contribution from viscous drag.
RAINEY: Very interesting. Our experience is that the steady drift force on a TLP, and the nonlinear tether tensions, are also strongly affected by viscous drag, in survival waves. Perhaps this vindicates our simplified slender-body approach to potential flow loads — its simplicity matches the modelling of viscous drag forces with a Morison drag term.