Some problems in the theory of hydrodynamical images, by F. Ursell, Department of Mathematics, Manchester University, Manchester M13 9PL, England, U.K.

Consider a potential flow field generated by source singularities in the infinite region outside a fixed body. The total velocity potential can then be expressed as the sum of the potentials due to the source singularities, together with a regular potential which is usually called the image potential of the singularities in the fixed body. (Similar considerations apply to other boundary conditions, to wave equations, and also to moving bodies.) The image potential can be found in various ways, in the general case by solving an integral equation, but for some simple shapes there are simpler methods, e.g. the method of separation of variables, or the method of multipoles, or integer transforms. For some shapes the image potential can be expressed even more simply as the sum of certain source potentials inside the body, i.e. outside the flow field. In some applications an inverse method is used: a suitable distribution of singularities is placed in the exterior field, and it is then found that there is a closed streamline so that the total potential represents the flow outside a certain fixed body (as well as a flow inside the fixed body); a wide class of fixed bodies can thus be generated. A well-known example is slender-body theory where a body in a uniform stream parallel to the axis is represented by sources along the axis. The direct method is helpful in the choice of suitable singularities in the inverse method, and it would be helpful if a dictionary of explicit images could be compiled.

Some well-known examples of the direct method are:

1) a source of strength m at a distance f from the centre of a hard sphere of radius $a < f$ on which the boundary condition is $\partial\phi/\partial r = 0$; the image system is a source of strength ma/f at the inverse point, together with a uniform line of sinks from the centre to the inverse point; this result is due to Hicks (1880), see Lamb, Hydrodynamics, §96. Let the potential of a source at a point $\mathbf{r} = \mathbf{f}$ outside a hard sphere be denoted by $G(\mathbf{r}, \mathbf{f})$, where bold-face symbols denote vector quantities, thus

$$G(\mathbf{r},\mathbf{f}) = \frac{m}{|\mathbf{r}-\mathbf{f}|} + \frac{ma/f}{|\mathbf{r}-a^2f^{-2}\mathbf{f}|} - ma/f\int_0^1 \frac{dv}{|\mathbf{r}-va^2f^{-2}\mathbf{f}|}.$$

Then it is not difficult to show that in the spherical shell $a^2/f < r < f$ (and therefore everywhere by analytic continuation) we have

$$G(\mathbf{r},\mathbf{f}) = ar^{-1}G(a^2r^{-2}\mathbf{r},\mathbf{f}) + \int_a^r G(Rr^{-1}\mathbf{r},\mathbf{f})dR, \tag{0.1}$$

The Hicks potential $G(\mathbf{r}, \mathbf{f})$ can be used to prove a result on analytic continuation. Suppose that the harmonic function ϕ is defined in a three-dimensional domain bounded by the surface S. Suppose that part of S is a surface S_a on which $r = a$, and that $\partial\phi/\partial r = 0$ on S_a. (Note that S_a need not be the complete sphere $r = a$). Suppose also that every radius from the origin intersects $S - S_a$ only once. Then ϕ can be continued across S_a into the image domain, the argument involves the Green's representation by Hicks sources and dipoles. The continuation can be found explicitly by using the functional relation (1.1) above:

$$\phi(\mathbf{r}) = ar^{-1}\phi(a^2r^{-2}\mathbf{r}) + \int_a^r \phi(Rr^{-1}\mathbf{r})dR,$$

where bold-face symbols represent vectors. The proof of this result (Weiss's Theorem) again uses Green's representation of ϕ by sources $G(\mathbf{r}, \mathbf{f})$ and dipoles $\partial/\partial n_f\, G(\mathbf{r}, \mathbf{f})$ over the surface S.

If the boundary condition on the sphere is $\phi = 0$ then the image is a sink of strength ma/f at the inverse point. Thus there are different images for different boundary conditions.

2) an ellipse or a spheroid in a uniform stream; here the total potential is the potential of the uniform stream, together with a single term in elliptic or spheroidal coordinates which is discontinuous or singular along the line joining the foci. The source or dipole strength can then be obtained from the discontinuity in the normal velocity or in the potential along the focal line or the focal ellipse. E.g. consider an ellipse in a uniform stream parallel to the major axis (Lamb, 1932 § 71). The flow is easily found in elliptic coordinates. Write $z = x + iy$, then the complex potential is found to be

$$w = Uz + Ub(a-b)^{-1}.\{z - (z^2 - c^2)^{1/2}\},$$

thus the image potential is

$$w_{\text{image}} = Ub(a-b)^{-1}.\{z - (z^2 - c^2)^{1/2}\},$$

where U is the velocity parallel to the x-axis, $2a$ is the major axis, $2b$ is the minor axis, and $c^2 = a^2 - b^2$. We see that the complex potential is defined both inside and outside the ellipse, with a cut from $z = -c$ to $z = +c$. Across the cut the potential is continuous and the normal velocity is discontinuous, thus the image potential can be interpreted as a line of sources,

$$w_{\text{image}} = \int_{-c}^{c} h(X) \log(z - X) dX,$$

where

$$h(x) = -\frac{U}{\pi} \frac{b}{a-b} \frac{x}{(c^2 - x^2)^{1/2}}.$$

3) an ellipsoid in a uniform stream; here the image potential is a single integral involving confocal coordinates. By analytic continuation into the ellipsoid this can be expressed as a distribution of singularities over the focal ellipse. The source or dipole strength is given by the discontinuity across the focal ellipse, as in the previous example. (See Miloh, 1974).

4) a two-dimensional acoustic wave symmetrically incident on a corner (equivalent to a three-dimensional water wave incident on a wedge); the acoustic velocity potential can be expressed as a complicated contour integral (Bowman, Senior, Uslenghi, 1969, ch.6). The action of the corner can be represented as due to a line distribution of acoustic sources along the line of symmetry inside the corner. The source strength can be found as follows: the acoustic potential can be continued explicitly across the two straight boundaries, the discontinuity in the normal velocity across the line of symmetry gives the source strength of the image. (The source must satisfy a radiation condition at infinity.) For small angles the source strength involves a Fresnel integral. This image has been applied in ship hydrodynamics by Faltinsen and Ogilvie.

5) Images of submerged two-dimensional multipoles. Here the image is the image in a free-surface condition and is usually obtained in the form of an integral convergent in the lower half-plane. This can be continued into the cut upper half-plane by deforming the contour of integration; for an application see Ursell (1968). The interpretation as a source distribution is rarely useful.

6) Images of three-dimensional multipoles in a cylindrical boundary. (This problem has recently arisen in the problem of acoustic trapped modes in a waveguide, see Ursell 1991, but arises in many other applications.) For simplicity consider a unit electrical charge on the axis of

an infinitely long earthed conducting cylinder of radius ℓ. It is not difficult to find the electrostatic potential everywhere inside the cylinder and in particular near the charge. Let cylindrical polar coordinates x, r be taken with the origin at the point charge and with the x-axis along the axis of the cylinder, then the electrostatic potential $\phi(x,r)$ satisfies Laplace's equation

$$\phi_{rr} + r^{-1}\phi_r + \phi_{xx} = 0 \text{ when } -\infty < x < \infty,\ 0 < r < l, \tag{6.1}$$

with the conditions

$$\phi(x,l) = 0 \text{ when } -\infty < x < \infty, \tag{6.2}$$

$$\phi(x,r) - (x^2 + r^2)^{-1/2} \text{ is bounded near } (x,r) = (0,0). \tag{6.3}$$

We now note that the products $K_0(kr)\cos kx$ and $I_0(kr)\cos kx$ are harmonic functions, where the functions $I_0(X)$ and $K_0(X)$ are the usual Bessel functions of imaginary argument; we also note that

$$(x^2 + r^2)^{-1/2} = \frac{2}{\pi}\int_0^\infty K_0(kr)\cos kx\,dk, \tag{6.4}$$

cf. Erdélyi,1953,§7.12,(27). It is now obvious that the potential

$$\phi(x,r) = (x^2 + r^2)^{-1/2} - \frac{2}{\pi}\int_0^\infty \frac{K_0(kl)}{I_0(kl)} I_0(kr)\cos kx\,dk \tag{6.5}$$

is the solution of our boundary value problem. The integral term in (6.5) is the image potential. The representation (6.5) is convergent inside and on the cylinder, in fact inside the larger cylinder $r = 2l$. From the modal expansion, valid for $x > 0$, it can be shown that the image of the point source in the cylindrical boundary is a distribution of ring sources over the plane through the source and normal to the axis; the source strength can in principle be obtained from an analytic continuation of the Fourier integral but no simple method has yet been discovered of continuing this integral. However, the modal expansion converges everywhere except on the mid-plane, and the source strength is then found as the sum (more precisely the Abel sum) of a slowly convergent series. The analytical expressions for the source strength take different forms in the annular regions $r/l = 2s + 1 + \sigma$, where $-1 < \sigma < 1$ and $s = 1, 2, 3, \ldots$, on the plane $x = 0$. The source strength is bounded on annuli on which s is even, but becomes unbounded on the limiting circles of those annuli on which s is odd. The image source strength vanishes on the annulus on which $s = 0$ and it may possibly vanish on all the other even annuli but this conjecture has not been proved or disproved. Our knowledge is still very incomplete. It is hoped to publish the details of this calculation elsewhere.

References

Bowman, J.J., Senior, T.B.A., Uslenghi, P.L.E. 1969 *Electromagnetic and acoustic scattering by simple shapes.* Amsterdam: North Holland.

Erdélyi, A. (ed.) 1953 *Higher transcendental functions.* Bateman Manuscript Project. New York: McGraw-Hill

Lamb, H. 1932, *Hydrodynamics*, 6th edn, Cambridge University Press.

Miloh, T. 1974 The ultimate image singularities for external ellipsoidal harmonics. SIMA Jl.App.Math. **26**, 334-344.

Ursell, F. 1968, The expansion of water-wave potentials at great distances. Proc.Camb.Phil.Soc., **64**, 1968, 811-826.

Ursell, F. 1991, Trapped modes in a circular cylindrical acoustic waveguide. Proc.Roy.Soc.Lond., **A.435**, 575-589.

Weiss, P. 1944 On hydrodynamical images: Arbitrary irrotational flow disturbed by a sphere. Proc.Camb.Phil.Soc. **40**, 259-261.