The Method of Modified Residue Calculus Applied to Horizontal Plate Problems in Linear Water Waves

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1 Introduction

The aim of this paper is to demonstrate the use of the Modified Residue Calculus method (hereafter referred to as the MRC method), in deriving rigorous solutions to various linear water wave problems.

For all the problems, we consider the depth of the water to be finite and constant, and that the plates are fixed rigidly, so as not to cause any radiation of waves. We also assume that the motion is periodic in time and that the waves interact obliquely with the plates. These assumptions simplify the problem into solving the modified Helmholtz equation in two dimensions instead of the three-dimensional Laplace equation.

The types of problems we can solve using this method are the scattering from semi-infinite plates either on the surface of the water or submerged, wave interaction with finite length plates, trapped waves within a gap of an infinite plate placed on the surface of the water, and trapped waves above a submerged finite length plate.

For the MRC method to work, all of the problems except the semi-infinite plate problems have to be symmetric about a centre line perpendicular to the surface of the water and sea-bed. This symmetry allows us to further simplify the finite plate/gap problems into half-plane problems.

The semi-infinite problems are well known problems and can be solved fairly easily by applying the Weiner-Hopf technique to them (see Heins [3, 4]), but we will resolve them using the MRC method. The reasons for this are firstly to demonstrate that the method works well and secondly to provide solutions, which we can later modify in order to solve the finite length plate/gap problems.

The trapped wave problems have analogous problems in acoustics which have been solved by Evans [1] and Evans, Linton & Ursell [2] using this same method. Also, this method has been previously employed by Mittra & Lee [6] and earlier by Jones [5] for similar problems in electromagnetic wave propagation.

2 Equations of Motion

Using the linear water-wave theory assumptions of an inviscid and incompressible fluid, and that the motion of the fluid is time-harmonic and periodic in the longshore direction, we seek solutions to the following conditions

$$(\nabla^2 - \ell^2)\phi = 0 \quad \text{in the fluid,} \tag{1}$$

$$K\phi = \frac{\partial \phi}{\partial y} \quad \text{on the free surface,} \tag{2}$$

where $K = \omega^2/g$,

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{on the sea bed,} \tag{3}$$

and

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{on the plate,} \tag{4}$$

where ω is the angular frequency, g is gravity, ℓ is the longshore wave number and $\phi(x, y)$ is the velocity potential.

3 Method of Solution

The first step is to take each problem and seperate it into regions. All the problems for which the plate is on the surface there will be two regions, one for when the water has a free surface, and the other for when the water is covered by a plate. The submerged plate problems give rise to three regions, one for the region below the plate, one for the region above the plate with a free-surface and finally for the region with just a free-surface.

Solutions in the form of eigenfunction expansions are then found for each region. The expansions are then matched at the regions common boundaries giving rise to one or two infinite systems of homogeneous linear equations depending whether you are solving for the surface plate or submerged plate problems. These infinite systems of equations are of the form

$$F_m + \sum_n^{\infty} U_n \left(\frac{1}{A_m - B_n} + \frac{C_m}{A_m + B_n} \right) = 0, \quad m \text{ an integer,} \tag{5}$$

and where U_m is unknown, F_m, A_m, B_m and C_m are known and C_m is exponentially decreasing with m and is equal to zero for the semi-infinite problems. Note that for the submerged plate case there will be two of these systems.

The next step in the MRC method is to generate a function $g(z)$ with poles at $z = A_m$ and zeros possibly at $z = B_m$, (because of a second infinite system in the submerged case we may also require zeros at $z = D_m$). Consider

$$H_m = \frac{1}{2\pi i} \int_{c_N} g(z) \left(\frac{1}{z - B_m} + \frac{C_m}{z + B_m} \right) \mathrm{d}z \tag{6}$$

where c_N is a curve which increases in size as N gets larger, and is chosen not to pass through any poles. By choosing $g(z)$ to have the correct behaviour at infinity we find that

$I_m \to 0$ as $N \to \infty$. So, using residue calculus on the integral we find

$$g(B_m) + C_m g(-B_m) + \sum_n \text{Res}(g; A_n) \left(\frac{1}{A_m - B_n} + \frac{C_m}{A_m + B_n} \right) = 0. \tag{7}$$

Now choosing

$$g(B_m) + C_m g(-B_m) = F_m \tag{8}$$

and

$$U_m = \text{Res}(g; A_m), \tag{9}$$

we can satisfy equation (5) and hence solve for U_m, and following on, solve for the velocity potential $\phi(x, y)$.

4 Conclusion

The method of Modified Residue Calculus has been used to solve a selection of linear water wave problems which all involve the interaction of obliquely incident waves with horizontal plates, either submerged or placed upon the surface of the water.

Exact solutions were obtained for the semi-infinite problems, which agree with those of Heins [3, 4], and rigorous solutions to the finite plate/gap problems are provided for the case when the plate/gap width is sufficiently *large*. This restriction occurs in the analysis, but, when numerically calculating information from the solutions, we find that the solutions are very convergent and can be used for when the plate/gap width is *small*.

References

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