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Resonant Reflection of Surface Waves Travelling Over Bottom Undulations

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1 Introduction

The reflection of surface waves by patches of bottom undulations has been extensively studied recently, owing to its importance in the development of shore-parallel bars. A straightforward perturbation analysis shows that large resonant reflection occurs under a Bragg condition (Davies 1982), namely when the bottom undulation has a wavelength half that of the incident waves. Laboratory experiments of Davies & Heathershaw (1984) confirm this prediction and suggest a possible practical application of this mechanism for the protection of beaches. For mild incident wave and bottom slopes, reflection at or near Bragg resonance is well predicted by perturbation theory based on multiple scales and the assumption of linearized surface waves (Mei 1985)

For moderate to large wave and/or bottom steepnesses, it is generally expected that higher-order Bragg resonance must also occur due to nonlinear interactions between the surface waves and bottom undulations. For a bottom containing unidirectional doubly-sinusoidal ripples, significant Bragg reflection corresponding to differences of the bottom ripple wavenumbers is observed in experiments even for small undulation amplitudes (Guazzelli *et al* 1992). This nonlinear resonant reflection can be comparable in magnitude (although not at the same frequency) to that due to linear Bragg effect. Since this phenomenon is a result of higher-order Bragg interactions, it *cannot* be predicted by the theory of Mei (1985).

In this work, we investigate nonlinear Bragg reflection of waves travelling over undulating bottom topography using an efficient new computational method. The method employs spectral (global) basis functions and numerically accounts for arbitrarily high order expansions in *both* the free surface steepness and bottom slope. For moderately steep slopes, the method obtains exponential convergence of the solution with both the number of spectral modes and the order of the expansions. Significantly, with the use of fast transform techniques, the computational burden is only linearly proportional to the number of modes and to the free surface and bottom perturbation orders. This powerful method is used to accurately predict resonant Bragg reflection by bottom undulations including high-order effects of the free surface and bottom.

2 Formulation

We consider surface wave propagation over a rippled bottom $z = -h + \zeta(x)$ with constant mean water depth h. Under the usual assumption of potential flow, the boundary-value problem for the velocity potential $\Phi(x, z, t)$ consists of Laplace's equation within the fluid, no flux condition on the bottom,

$$\Phi_z = \zeta_x \Phi_x \, , \qquad \text{on} \qquad z = -h + \zeta(x) \, , \tag{1}$$

and nonlinear kinematic and dynamic boundary conditions on the free surface. For initial conditions, the free surface elevation $\eta(x, 0)$ and velocity potential $\Phi(x, z, 0)$ are prescribed.

In terms of the surface potential $\Phi^S(x,t) = \Phi(x, \eta(x,t), t)$, we can write the free surface boundary conditions in Zakharov form as:

$$\eta_t + \Phi_x^S \eta_x - (1 + \eta_x^2)\Phi_z(x, \eta, t) = 0 \, , \quad \text{and} \tag{2}$$

$$\Phi_t^S + g\eta + \frac{1}{2}(\Phi_x^S)^2 - \frac{1}{2}(1 + \eta_x^2)\Phi_z^2(x, \eta, t) = 0 \, , \tag{3}$$

where g is the gravitational acceleration. In practice, these nonlinear equations can be considered as evolution equations for η and Φ^S, provided that the surface vertical velocity $\Phi_z(x, \eta, t)$ can be obtained from the boundary-value problem.

The present approach was first suggested by Dommermuth & Yue (1987). We consider regular perturbation expansions in both the bottom undulation ($\zeta(x)$) and the instantaneous free surface ($\eta(x,t)$) simultaneously. For simplicity (and without loss of generality), we assume that $\epsilon \ll 1$ measures both the bottom and free-surface wave slopes. Our intention is to solve the problem (in the time domain) to *arbitrary* high order M in ϵ and we write Φ in a perturbation series to order M:

$$\Phi(x, z, t) = \sum_{m=1}^{M} \Phi^{(m)}(x, z, t). \tag{4}$$

To obtain the free-surface and bottom boundary conditions for each $\Phi^{(m)}$ in (4), we expand (1) and Φ^S in separate Taylor series with respect to the mean surfaces $z = -h$ and 0 respectively. Collecting terms at the respective orders, we finally obtain a sequence of Neumann boundary conditions on $z = -h$:

$$\Phi_z^{(1)}(x, -h, t) = 0,$$

$$\Phi_z^{(m)}(x, -h, t) = \sum_{\ell=1}^{m-1} \left[\frac{\zeta^\ell}{\ell!} \frac{\partial^{(\ell-1)}}{\partial z^{(\ell-1)}} \Phi_x^{(m-\ell)}(x, -h, t) \right]_x \, , \qquad m = 2, 3, \ldots, M \, ; \tag{5}$$

and a sequence of Dirichlet boundary condition on $z = 0$:

$$\Phi^{(1)}(x,0,t) = \Phi^s,$$
$$\Phi^{(m)}(x,0,t) = -\sum_{\ell=1}^{m-1} \frac{\eta^\ell}{\ell!} \frac{\partial^\ell}{\partial z^\ell} \Phi^{(m-\ell)}(x,0,t), \qquad m = 2,3,\ldots,M\,. \tag{6}$$

At each order m, $\Phi^{(m)}$ satisfies Laplace's equation in the *mean* fluid domain $-h < z < 0$, the Neumann boundary condition (5) on $z = -h$, and the Dirichlet boundary condition (6) on $z = 0$. In a spectral approach, we represent $\Phi^{(m)}$ in terms of global basis functions which satisfy the field equation and the homogenous surface and bottom conditions. To accomplish this we write $\Phi^{(m)} = \alpha^{(m)} + \beta^{(m)}$, where

$$\alpha^{(m)}(x,z,t) = \sum_{n=0}^{\infty} \alpha_n^{(m)}(t) \frac{\cosh[|k_n|(z+h)]}{\cosh(|k_n|h)} e^{ik_n x} + \text{c.c.}\,, \tag{7}$$

$$\beta^{(m)}(x,z,t) = \beta_0^{(m)} z + \sum_{n=1}^{\infty} \beta_n^{(m)}(t) \frac{\sinh(|k_n|z)}{|k_n|\cosh(|k_n|h)} e^{ik_n x} + \text{c.c.}\,. \tag{8}$$

In the above, k_n is the wavenumber, *c.c.* denotes complex conjugate of the preceding term, and a 2π-periodic computational domain in x is assumed. Clearly, for smooth (periodic) $\Phi^{(m)}$, the amplitudes of the orthogonal spectral modes $\alpha_n^{(m)}$ and $\beta_n^{(m)}$ decay exponentially with increasing wavenumber k_n. At each order m, $\alpha_n^{(m)}$ and $\beta_n^{(m)}$ are determined by taking the inner product of $e^{ik_n x}$ with (6) and (5) respectively and the nonlinear boundary-value problem is solved sequentially for $m = 1,2,\ldots,M$. The overall problem is integrated in time via (2) and (3) starting from initial conditions.

3 Results and Discussions

To illustrate the efficacy of the present method, we first consider a single wave propagating over a horizontal bottom with uniformly sinusoidal ripples. This simple case has been well-studied both theoretically and experimentally. Nonlinear results near the Bragg resonance are obtained and compared to experiments (Davies & Heathershaw 1984), perturbation theory (Mei 1985), and the boundary-integral-equation numerical solution of Dalrymple & Kirby (1986). As an extension, we investigate the case of a doubly sinusoidal bed and obtain the subharmonic resonant reflection coefficient which we compare to the measurements of Guazzelli *et al* (1992).

Finally, for a bottom containing disordered ripples, localization of surface waves has been predicted by theory (Devillard *et al* 1988) and observed in experiments (Belzons *et al* 1988). The present efficient method is able to accurately simulate this localization phenomenon and provide a verification for the physical experiments.

4 References

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