An Efficient Numerical Method for 3-D Flow around a Submerged Body

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1 Introduction

The subject of this paper is an efficient, Schwartz–type, iterative method for computing free surface flows. In particular, the steady 3–D potential flow around a submerged body moving in a liquid of finite constant depth is considered. The motion is described in Cartesian coordinates which are fixed with respect to the body. The x-axis points opposite to the forward direction, the y-axis points sideways and the z-axis is directed vertically upwards. Let the depth of the liquid be d, the speed of the body be U and the acceleration of gravity be g. The physical quantities are scaled by the length d and the velocity $\sqrt{gd}$. We split the velocity potential, Φ, into a free stream potential and a perturbation potential, $\Phi = \mu x + \phi$, here $\mu = U/\sqrt{gd}$ is the Froude number.

To construct an efficient iterative method to solve this indefinite problem, we decompose it into two, mathematically simpler subproblems. Then we iterate between these subproblems and at convergence, the solution to the original problem is given as the sum of the solutions to the subproblems.

This problem can be solved by several existing techniques, eg. the boundary integral method described in [2] or the hybrid finite element method in [4]. The aim of the research described here is to take a step towards solving the problem where effects of vorticity an viscosity are taken into account in the region close to and in the wake of the submerged body. Another important reason was an accurate solution of the 3–D nonlinear potential problem, where the boundary integral method is known to depend on the addition of artificial dissipation at the free surface boundary.

2 The Subproblems

The perturbation potential is governed by:

$$\Delta\phi = 0, \; -\infty < x < \infty, \; -\infty < y < \infty, \; -1 < z < 0, \tag{1}$$

together with the boundary conditions

$$\begin{aligned} \mu^2\phi_{xx} + \phi_z &= 0, \; -\infty < x < \infty, \; -\infty < y < \infty, \; z = 0, \\ \phi_z &= 0, \; -\infty < x < \infty, \; -\infty < y < \infty, \; z = -1, \\ \partial\phi/\partial n + \mu\cos\theta &= 0, \text{ on the body.} \end{aligned} \tag{2}$$

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Here, $\partial/\partial n$ denotes the outward normal derivative and θ is the angle between the normal and the x-axis. We are looking for a solution where the perturbation potential tends to zero at large distances in front of the body. This condition is called the upstream condition,

$$\lim_{x\to-\infty} \phi = 0, \; -\infty < y < \infty, \; -1 < z < 0, \tag{3}$$

For the two-dimensional counterpart to the present problem, the most efficient solution approach is probably a direct method, cf. [5]. However, due to memory and work requirements, it is not possible to use a direct method to solve the three dimensional problem. The problem (1–3) is indefinite, which makes the convergence of most iterative methods unstable. To circumvent this difficulty, we decompose the problem into two more easily solvable subproblems and form a Schwarz–type iteration between these subproblems to solve the original problem.

The first subproblem, which will be refered to as the definite subproblem, is defined by

$$\Delta\phi^{I} = 0, \; -\infty < x < \infty, \; -\infty < y < \infty, \; -1 < z < 0. \tag{4}$$

together with the boundary conditions

$$\begin{aligned} \phi^{I}_{z} &= 0, \; -\infty < x < \infty, \; -\infty < y < \infty, \; z = 0, & (5)\\ \phi^{I}_{z} &= 0, \; -\infty < x < \infty, \; -\infty < y < \infty, \; z = -1, & (6)\\ \partial\phi^{I}/\partial n &= h, \text{ on the body.} & (7) \end{aligned}$$

The second subproblem, which will be called the indefinite subproblem, does not have a submerged body in the interiour of the domain. It is governed by,

$$\Delta\phi^{II} = 0, \; -\infty < x < \infty, \; -\infty < y < \infty, \; -1 < z < 0, \tag{8}$$

subject to the boundary conditions

$$\begin{aligned} \mu^2\phi^{II}_{xx} + \phi^{II}_{z} &= t, \; -\infty < x < \infty, \; -\infty < y < \infty, \; z = 0, & (9)\\ \phi^{II}_{z} &= 0, \; -\infty < x < \infty, \; -\infty < y < \infty, \; z = -1, & (10) \end{aligned}$$

To fix the undetermined constant in the subproblems we enforce upstream conditions similar to (3).

The first subproblem is definite and can therefore be solved by standard iterative methods, see §5. The second subproblem is indefinite but has no body. It is therefore easily and efficiently solvable by separation of variables, see §6.

3 The Iterative Cycle

The solutions of the subproblems are well defined once the forcing functions h and t are determined. It is clear that $\phi^{I} + \phi^{II}$ will solve (1–3) if we can find functions t and h that simultaneously satisfy $t(x,y) = -\mu^2\phi^{I}_{xx}(x,y,0)$ and $h(x_b,y_b,z_b) = -\mu\cos\theta(x_b,y_b,z_b) - \partial\phi^{II}/\partial n(x_b,y_b,z_b)$, where the boundary of the body is described by $x_b = x_b(s,u)$, $y_b = y_b(s,u)$, $z_b = z_b(s,u)$, $0 \le s \le 1$, $0 \le u \le 1$. We compute h and t by iteration. The initial guess is taken to be $\phi^{II(0)}(x,y,z) \equiv 0$ and iterate according to

1. Set $h^{(i)}(x_b,y_b,z_b) = -\mu\cos\theta(x_b,y_b,z_b) - \partial\phi^{II(i-1)}/\partial n(x_b,y_b,z_b)$ and solve the definite subproblem for $\phi^{I(i)}$.

2. Set $t^{(i)}(x,y) = -\mu^2\phi^{I(i)}_{xx}(x,y,0)$, and solve the indefinite subproblem for $\phi^{II(i)}$.

We have proven that the iteration converges for sufficiently small Froude numbers. In order to demonstrate the convergence numerically, we truncate the infinite domain in the x- and the y-direction and introduce farfield boundary conditions to carry out the practical computation. Finally in §7 we present numerical results for a second order accurate discretization of (1–3). We show that the iterative method converges rapidly. We also verify numerically that the convergence rate is essentially independent of the grid size.

4 Truncation of the computational domain

To limit the computational work we truncate the computational domain in the x- and y-direction to: $-b < x < b$, $-a < y < a$, where a, b are positive. For both subproblems we enforce solid wall boundary conditions for $y = \pm a$, ie. we are considering flow in a canal. For the definite subproblem we apply approximate in and outflow bc's at $x = \pm b$ and for the indefinite subproblem we prescribe exact in and outflow bc's at $x = \pm b$.

5 Solving the definite subproblem

We discretize the definite subproblem by second order accurate central finite differences using a composite overlapping grid. To apply this technique, we divide the domain into simple overlapping subdomains and cover each subdomain with a component grid. The subdomains attaching to the body are covered with bodyfitted curvilinear grids and the surrounding sea is covered with one or several Cartesian grids. The main advantage with this method compared to discretizing the whole domain with one single grid is that each component grid can be made logically rectangular and without singularities. The component grids can be constructed almost independently of eachother. The restrictions are that the component grids need to overlap eachother sufficiently where they meet and the union of the component grids have to cover the whole computational domain. The gridfunctions on the component grids are coupled by continuity requirements, which are enforced by applying sufficiently accurate, in this case quadratic, interpolation relations between the gridfunctions at the interiour boundaries where the component grids overlap. We use the fortran software package CMPGRD to construct the composite grids. A comprehensive description of this approach for a related problem is given in [5],[6].

The resulting linear system of equations is solved with the BCG method, using the CGES solver. This method requires of the order $\mathcal{O}(n^{4/3})$ operations where n equals the number of used gridpoints in the composite grid.

This subproblem could easily be solved using boundary integral methods and a fast iterative solver.

6 Solving the indefinite subproblem

To solve the indefinite problem (8–10) we use Fourier transform in the y-direction, then we separate x and z variables. This gives us a number of second order linear ODE's in x which we discretize using second order accurate central finite differences.

The character of the solution to the definite problem is smooth and local, whereas the solution to the indefinite problem contains downstream waves with a relatively small wavelength. Therefore, we utilize a cartesian grid with finer gridstep that covers a larger x-interval to compute the solution of the indefinite subproblem. The occuring tridiagonal systems of equations are solved by the subroutine DNBSL in the SLATEC package. The work needed to obtain a solution to the indefinite subproblem is of the order $\mathcal{O}(n_1)$, where n_1 is the product of the number of gridpoints in the discretization of one ordinary differential equation and the number of terms we retain in the series expansion. This n_1 is of the same order as the number of unknowns in the definite subproblem.

7 Numerical examples

To validate the iterative method we have studied a number of test cases. As a test body, we used a sphere with radius 1/6. The center of the sphere was submerged 0.5 below the free surface. The Froude number for these computations is: $\mu = 0.4$ unless anything else is mentioned.

To check the implementation of the method and see that the discretization error is of second order, we compare the computed discrete solution for three different gridsteps $3h, 2h$ and h in table 1, the values are normalized by $|\phi_h|_\infty, |\phi_h|_{RMS}$. We clearly see that the solution is second order accurate. (RMS here denotes the usual root-mean-square norm ie. the discrete L_2 norm)

In table 2 we show the number of iterations needed to decrease the relative increment in each step of the iteration to $1.E-05$ for the gridsteps $h - 2\sqrt{2}h$. We see that the convergence rate is only weakly dependent of the gridsize.

	Absolute Norm	RMS Norm
$\phi_{3h} - \phi_h$	1.3E-01	1.7E-02
$\phi_{2h} - \phi_h$	4.9E-02	6.5E-03
acc. order	2.0	2.0

Table 1: Accuracy test, the differences are normalized by $|\phi_h|_\infty$

Froude Number (μ)	C_D Havelock	C_D present
0.40	3.5E-02	3.2E-02
0.45	4.8E-02	4.5E-02
0.50	5.8E-02	5.7E-02
0.55	6.0E-02	5.9E-02

Table 3: Coefficient of drag ,C_D, comparison

gridstep	nr. of iterations
$2\sqrt{2}h$	7
$2h$	7
$\sqrt{2}h$	8
h	9

Table 2: Number of iterations for convergence for different gridsteps

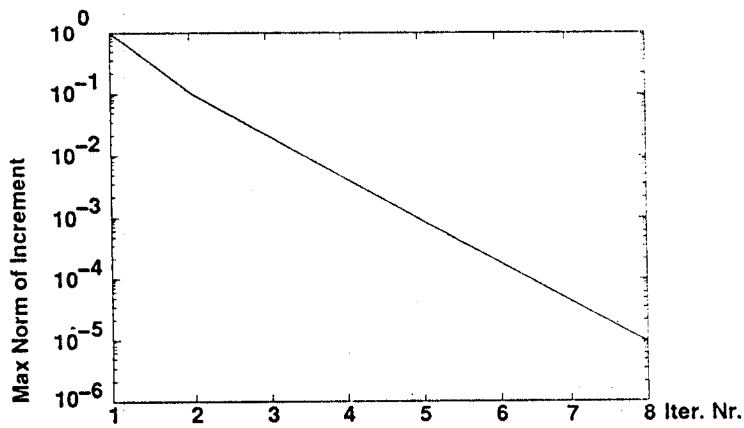

Figure 1: Relative increment at each iteration ($\mu = 0.4$).

Furthermore, in figure 1 we show the max-norm of the relative increment at each iteration. We clearly see that the convergence is linear and fast.

To ensure that the iterative method gives the same solution as other methods, we compare the coefficient of drag (C_D) with computations made by Havelock [3], 1931, in table 3 where he computes the flow around a submerged sphere. The coefficient of drag is defined by:

$$C_D = \oint_{\text{body}} (2\phi_x + \phi_x^2 + \phi_y^2 + \phi_z^2)\, dS$$

We see that there is good agreement with the previously computed C_D's.

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