Nonlinear wave forces on offshore structures

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Introduction

The need to include nonlinear terms when calculating the wave loading on offshore structures is well-established, as there are many important effects which cannot be predicted by linear theory alone. In particular, many authors have considered the numerically nontrivial task of extending linear theory to second order. The difficulty which arises with this approach occurs because the second-order potential satisfies an inhomogeneous free surface boundary condition. Whether this potential is solved for directly, (see eg Kim & Yue 1988), or simply the second-order forces determined by introducing an auxiliary potential, (see eg Molin 1979), the resulting calculation usually involves the computation of a slowly decaying oscillatory integral over the whole of the free surface. The purpose of this work is to show that for fixed, two-dimensional bodies in monochromatic waves this computational work may be reduced by writing at least part of the second-order potential in terms of harmonic functions which are constructed from products of derivatives of the first-order potential.

Theoretical analysis

A wave is incident from the left on a two-dimensional, fixed convex body which intersects the mean free surface at right angles. The fluid is assumed to have infinite depth and coordinate axes are chosen such that the x-axis is horizontal and the y-axis points vertically downwards, with the origin positioned so that the body intersects the free surface at the points $(-a, 0)$ and $(a, 0)$. The wave steepness ϵ is assumed to be small and the velocity potential is expanded in a power series in ϵ. At second-order there is both a steady contribution to the potential and a contribution at twice the frequency of the incident wave. Vada (1987) showed that the double frequency potential $Re[A^2\omega\phi_2(x, y)\exp(-2i\omega t)]$ satisfies the inhomogeneous free surface condition

$$4K\phi_2 + \frac{\partial\phi_2}{\partial y} = \frac{i}{2K}\left[3K^2\phi_1^2 + 2\left(\frac{\partial\phi_1}{\partial x}\right)^2 + \phi_1\frac{\partial^2\phi_1}{\partial x^2}\right], \quad \text{on} \quad y = 0, \quad |x| > a, \qquad (1)$$

in the usual notation, where $Re[-igA\phi_1\exp(-i\omega t)/\omega]$ is the first-order potential. By interpreting the right-hand side of (1) as a pressure distribution over the free surface, the method of Wehausen & Laitone (1960) may be used to generate a function which is harmonic everywhere in the region $y > 0$ and satisfies the inhomogeneous boundary condition (1). The second-order potential is then constructed from a combination of this function and a harmonic function which satisfies the homogeneous free-surface condition and is chosen so that the sum of the two functions satisfies the condition of no-flow through the body. However, this approach does not make use of the specific form of the right-hand side of (1) and it is here shown that part of the particular solution for ϕ_2 may be constructed from harmonic functions which are products of first-order quantities. This alternative method allows the vortex-like behaviour of ϕ_2 at large depths, derived by Newman (1990), to appear explicitly in the expression for ϕ_2.

It is convenient to introduce the first-order complex velocity

$$w'(x+jy) = \frac{\partial\phi_1}{\partial x} - j\frac{\partial\phi_1}{\partial y} \tag{2}$$

and the square of this function

$$(w')^2 = \left(\frac{\partial\phi_1}{\partial x}\right)^2 - \left(\frac{\partial\phi_1}{\partial y}\right)^2 - 2j\frac{\partial\phi_1}{\partial x}\frac{\partial\phi_1}{\partial y}. \tag{3}$$

(The complex number j is used here to denote the square root of '-1' to avoid confusion with the use of i in the time dependence.) It is straightforward to show, from the theory of an analytic functions, that

$$\alpha = \left(\frac{\partial\phi_1}{\partial x}\right)^2 - \left(\frac{\partial\phi_1}{\partial y}\right)^2 \tag{3}$$

and

$$\beta = 2\frac{\partial\phi_1}{\partial x}\frac{\partial\phi_1}{\partial y} \tag{4}$$

are harmonic functions. Furthermore, α satisfies the free surface condition

$$4K\alpha + \frac{\partial\alpha}{\partial y} = -4K^3\phi_1^2 + 2K\left(\frac{\partial\phi_1}{\partial x}\right)^2 - 2K\phi_1\frac{\partial^2\phi_1}{\partial x^2}, \quad \text{on} \quad y=0, \quad |x|>a \tag{5}$$

and β satisfies the condition

$$4K\beta + \frac{\partial\beta}{\partial y} = \frac{\partial}{\partial x}\left[-3K^2\phi_1^2 - \left(\frac{\partial\phi_1}{\partial x}\right)^2\right], \quad \text{on} \quad y=0, \quad |x|>a. \tag{6}$$

From a comparison of the right-hand sides of (1), (5) and (6), it is clear that the function α may be used in the construction of a particular solution for ϕ_2, but it is more appropriate to consider an integral of β. Thus, the harmonic function γ is defined by

$$\gamma = Im_j\left[\int_{-L+j0}^{z}(w')^2\,dz'\right] + K\phi_1^2(-L,0) - 2KR, \tag{7}$$

where R is the first-order reflection coefficient, $Im_j[(a+ib)+j(c+id)] = c+id$, $z' = -L+j0$ is a point on the free surface to the left of the body and the contour of integration is horizontally along the free surface from the point $z' = -L$ and then vertically downwards to the point z, passing around part or all of the body contour as necessary. After some manipulation, γ may be written as

$$\gamma(x,y) = K\phi_1^2(x,0) + \int_0^y \left(\frac{\partial\phi_1}{\partial x}\right)^2 - \left(\frac{\partial\phi_1}{\partial y'}\right)^2 dy' - 2KRH(-x), \quad |x|\geq a \tag{8}$$

for points (x,y) which are not below the body, where $H(-x)$ is the Heaviside function and an application of Cauchy's theorem has been used to express an integral around the body in terms of far-field quantities. It is straightforward to show that γ satisfies the free surface condition

$$4K\gamma + \frac{\partial \gamma}{\partial y} = 3K^2\phi_1^2 + \left(\frac{\partial \phi_1}{\partial x}\right)^2 - 8K^2RH(-x) \quad \text{on} \quad y=0, \quad |x|>a \tag{9}$$

Motivated by the form of the right-hand side of (9), it is convenient to introduce the harmonic function

$$\lambda = \frac{1}{4\pi K}\left(\theta - \frac{\pi}{2}\right) + \frac{1}{16\pi K^2}\frac{\sin\theta}{r}, \tag{10}$$

where $x = r\sin\theta$ and $y = r\cos\theta$. This function is derived from Ursell's wave free antisymmetric potentials (1949) and satisfies the free surface condition

$$4K\lambda + \frac{\partial \lambda}{\partial y} = -H(-x), \quad \text{on} \quad y=0, \quad x \neq 0. \tag{11}$$

Thus, from (5), (9) and (10), ϕ_2 may be expressed as

$$\phi_2 = \frac{i}{4K^2}\left[\left(\frac{\partial \phi_1}{\partial x}\right)^2 - \left(\frac{\partial \phi_1}{\partial y}\right)^2\right] + \frac{i}{2K}\gamma - \frac{iR}{\pi}\left(\theta - \frac{\pi}{2}\right) - \frac{iR}{4\pi K}\frac{\sin\theta}{r} + \chi_2 \tag{12}$$

where χ_2 is a harmonic function which satisfies the free surface condition

$$4K\chi_2 + \frac{\partial \chi_2}{\partial y} = \frac{i\phi_1}{K}\left[K^2\phi_1 + \frac{\partial^2 \phi_1}{\partial x^2}\right] \quad \text{on} \quad y=0, \quad |x|>a. \tag{13}$$

The specific combination of α, γ and λ is subtracted out from ϕ_2 so that the remaining potential χ_2 satisfies an inhomogeneous free surface condition, the right-hand side of which decays to zero as $|x| \to \infty$. It is the nondecaying forcing on the free surface which produces the leading order behaviour of ϕ_2 at large depths, shown to be vortex-like by Newman (1990). As α and γ decay to zero as $y \to \infty$, this dominant behaviour appears explicitly in the representation of ϕ_2 in (12) as a multiple of the line vortex potential λ.

Discussion

The representation for ϕ_2 in (12) does not yield a particular solution for ϕ_2 which satisfies the total inhomogeneous free surface boundary condition in (1). However, further progress may be made if ϕ_1 may be expressed as an expansion in multipole potentials. In this case, ϕ_1 satisfies

$$\phi_1 = e^{iKx} + \sum_{m=2}^{\infty}\frac{a_m}{x^m} + a_0 \int_0^{\infty}\frac{\cos kx}{k-K}\,dk + a_1 \lim_{y \to 0}\int_0^{\infty}\frac{k e^{-ky}\sin kx}{K(k-K)}\,dk \text{ on } y=0,\ |x| \geq a \tag{14}$$

and

$$K^2\phi_1 + \frac{\partial^2 \phi_1}{\partial x^2} = \sum_{m=2}^{\infty} \frac{b_m}{x^m} \quad \text{on} \quad y = 0, \quad |x| > a, \tag{15}$$

where a_m and b_m are constants. (The series in (15) may not converge on $y = 0$ at $x = a$ as there is a possible logarithmic singularity in $\partial^2\phi_1/\partial x^2$ at this point.) For illustrative purposes, it is convenient to consider the construction of a harmonic function u which satisfies

$$4Ku + \frac{\partial u}{\partial y} = \epsilon^{iKx} \sum_{m=2}^{N} \frac{b_m}{x^m} \quad \text{on} \quad y = 0, \quad |x| > a \tag{16}$$

and decays as $y \to \infty$. In order to construct such a function, it is instructive to observe that if $f(x + iy)$ is an analytic function of $x + iy$ and $g(x - iy)$ is an analytic function of $x - iy$ then the function $f(x + iy) + g(x - iy)$ is harmonic. Thus, it may be shown by substitution in (16), that a particular solution for u is given by

$$u = \epsilon^{iK(x+iy)} \sum_{m=1}^{N-1} \frac{c_m}{(x + iy)^m} + 3iKc_1\epsilon^{iK(x+iy)} \int_0^{\infty} \frac{\epsilon^{3ik(x+iy)}}{K - k} dk, \tag{17}$$

where the coefficients c_m satisfy the recurrence relation

$$3Kc_m - i(m-1)c_{m-1} = b_m, \qquad c_{N-1} = ib_N/(N-1). \tag{18}$$

Numerically, the coefficients c_m should be determined using backward recurrence to ensure that they do not become too large. Similar procedures may be adopted to generate harmonic functions which satisfy other relevant free surface boundary conditions and the choice of whether the function should depend on $x + iy$ or $x - iy$ or a mixture of both is determined by the need to ensure that the functions decay as $y \to \infty$. This work is ongoing.

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