# DEFORMABLE FLOATING BODIES

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## INTRODUCTION

The motions of floating bodies usually are described by six rigid-body modes including translations (surge, sway, heave) parallel to the Cartesian $(x, y, z)$ axes, and (roll, pitch, yaw) rotations about the same axes. These modes are denoted by the indices $(j = 1, 2, 3, 4, 5, 6)$, respectively. In the linear analysis each mode can be defined uniquely by a (small) time-dependent amplitude $\xi_j(t)$.

The same notation can be extended to higher-order modes of body deformation $(j = 7, 8, ...)$. These may include continuous structural deflections, discontinuous deflections such as the angular deformation of a hinged vessel, or the motions of an array of separate bodies which are analyzed collectively as a single 'global body'. Another application is to represent wall effects in a wave tank with a truncated array of images; depending on the mode of motion of the original body, each image must move in a symmetric or antisymmetric manner which can be described by a suitable higher-order mode of the global array. In heave, where all of the images are in phase with the body, the entire global array moves in phase as if it were a single rigid body; but in sway (transverse to the tank axis) the images must alternate in sign, requiring a non-rigid global motion for the array.

In general, each mode may be defined by a vector 'shape function' $\mathbf{S}_j(\mathbf{x})$, with Cartesian components $(u_j, v_j, w_j)$. The displacement of an arbitrary point within the body, due to the corresponding mode, is represented then by the product $\xi_j(t)\mathbf{S}_j(\mathbf{x})$. The vector $\mathbf{S}_j$ is assumed to be continuous and differentiable near the body surface $S_b$, with divergence $D_j = \nabla \cdot \mathbf{S}_j$. The divergence is zero for each rigid-body mode. The normal component of $\mathbf{S}_j$ on $S_b$ is expressed in the form

$$n_j = \mathbf{S}_j \cdot \mathbf{n} = u_j n_x + v_j n_y + w_j n_z \tag{1}$$

The unit normal vector $\mathbf{n}$ points out of the fluid domain, and into the body.

Corresponding to these modes of motion, generalized pressure forces are defined in the form

$$F_i = \iint_{S_b} p n_i dS = -\rho \iint_{S_b} \left( \frac{\partial \phi}{\partial t} + \tfrac{1}{2} V^2 + gz \right) n_i dS \tag{2}$$

Here $p$ is the fluid pressure, $\phi$ the velocity potential, and $V$ the scalar fluid velocity. In the first-order analysis the generalized force includes contributions from the added mass, damping, wave diffraction, and hydrostatic pressure. In some cases second-order forces may be of interest as well. Each of these is considered in the following Sections.

## FIRST-ORDER HYDRODYNAMIC PRESSURE FORCE

The linear description of the generalized velocity potential and resulting pressure force is a trivial extension of the rigid-body analysis. It is convenient here to assume time-harmonic motion, with the unsteady mode amplitude factor $\xi_j(t)$ replaced by the real part of $\xi_j e^{i\omega t}$. Corresponding to each mode there is a radiation potential $\xi_j \phi_j(\mathbf{x})$, distinguished by the boundary condition $\partial\phi_j/\partial n = i\omega n_j$ on $S_0$, the mean position of $S$. Added-mass and damping matrices are defined in the form

$$\omega^2 a_{ij} - i\omega b_{ij} = -i\omega\rho \iint_{S_0} \phi_j n_i dS = -\rho \iint_{S_0} \phi_j \frac{\partial \phi_i}{\partial n} dS \tag{3}$$

and the generalized wave-exciting force is

$$X_i = -i\omega\rho \iint_{S_0} \phi_d n_i dS = -\rho \iint_{S_0} \phi_d \frac{\partial \phi_i}{\partial n} dS \tag{4}$$

Here $\phi_d$ is the diffraction (incident plus scattered) potential. The indices $i$ and $j$ can take on any values within the ranges of the rigid-body modes (1-6) and extended modes (7,8,...).

Green's theorem can be applied to (3) to establish reciprocity, and to (4) to derive the Haskind relations between the generalized exciting force and the corresponding radiation potential.

## FIRST-ORDER HYDROSTATIC FORCE

The deformation of the body geometry must be considered in deriving the contribution to the generalized force (2) from the hydrostatic pressure $-\rho g z$, since this quantity is of order one. Three surfaces of integration are defined as follows: The initial wetted surface of the body prior to the modal displacement $\mathbf{S}_j$ is denoted as $S_0$, and the corresponding deformed surface after the normal displacement in the mode $j$ is denoted as $S_\delta$. Both are open surfaces, with their upper boundaries in the plane $z = 0$. The closed surface $\Sigma$ is defined including $S_0$, $S_\delta$, and the portion of the plane $z = 0$ lying between these two open surfaces. On $\Sigma$ the normal vector is defined in a consistent manner, to point out of the enclosed volume $\mathcal{V}$.

With these definitions, the change in the hydrostatic generalized force component $F_i$ due to a unit displacement of the body in mode $j$ is defined by the matrix

$$C_{ij} = \rho g \iint_{S_\delta} z n_i dS - \rho g \iint_{S_0} z n_i dS = \rho g \iint_{\Sigma} z n_i dS \tag{5}$$

Using (1) and the divergence theorem,

$$C_{ij} = \rho g \iint_{\Sigma} z\mathbf{S}_i \cdot \mathbf{n} dS = \rho g \iiint_{\mathcal{V}} \nabla \cdot (z\mathbf{S}_i) dV \tag{6}$$

For small deformations the volume $\mathcal{V}$ is thin, and the last integral can be approximated to first order as the surface integral of the product of the integrand and the distance $n_j$ between the two boundary surfaces $S_0$ and $S_\delta$. Thus

$$C_{ij} = \rho g \iint_{S_0} n_j \nabla \cdot (z\mathbf{S}_i) dS = \rho g \iint_{S_0} n_j (w_i + zD_i) dS \tag{7}$$

Here the generalized force is defined in a fixed reference frame, and only the hydrostatic pressure is considered. As a result, (7) includes some contributions which normally are not considered, such as the roll (or pitch) moment due to a sway (or surge) displacement, both of which are equal to the displaced body volume times the moment arm associated with the corresponding displacement. Normally, for a freely floating body, these contributions are balanced by the gravitational force due to the body mass. In the generalized analysis of a deformable body the corresponding mass force must be evaluated separately, for each mode, depending on the mode shape and mass distribution. The coefficients (7) depend not only on the normal displacement (1), but also on the divergence $D_i$ and the vertical component of $\mathbf{S}$.

As a simple example consider a floating circular cylinder, of radius $r_0$ about the $x$-axis. The only nontrivial rigid-body mode is heave and the relevant parameters in (7) are $w_3 = 1$, $D_3 = 0$, and $n_3 = \cos\theta$, in terms of polar coordinates such that $iy - z = re^{i\theta}$. Integrating over the submerged surface $(-\pi/2 < \theta < \pi/2)$ on $r = r_0$ gives the usual restoring coefficient $C_{33} = 2\rho g r_0$ per unit length along the $x$-axis. In addition, define a dilating mode ($i = 7$) where the shape function $\mathbf{S}_7$ is a unit vector in the radial direction. The divergence of this vector is $D_7 = 1/r$, $w_7 = -\cos\theta$, and $n_7 = -1$. From (7) it follows that $C_{77} = 4\rho g r_0$, and $C_{37} = C_{73} = -\pi\rho g r_0$. For a more general mode shape ($i = 7$) where the radial deformation is $f(\theta)$ the corresponding results are

$$C_{37} = -\rho g r_0 \int_{-\pi/2}^{\pi/2} f(\theta) d\theta, \quad C_{73} = -2\rho g r_0 \int_{-\pi/2}^{\pi/2} f(\theta)\cos^2\theta d\theta, \quad C_{77} = 2\rho g r_0 \int_{-\pi/2}^{\pi/2} f(\theta)\cos\theta d\theta$$

In general $C_{ij} \neq C_{ji}$.

## SECOND-ORDER FORCES

Substantial difficulties can be anticipated in evaluating the second-order contributions to (2), following the direct approach outlined by Ogilvie (1983) and also by Lee & Newman (1991). Particular attention must be given to the transfer of the first-order pressure between the exact and mean body positions. The following approach avoids this complication.

Define $S_{bf}$ as the union of the (exact) body surface and the portion of the free surface inside a fixed control surface $S_c$, which is vertical near $z = 0$ and intersects the free surface $z = \varsigma$ along the contour $C_c$. The continuous differentiable shape function $\mathbf{S}_i$ is assumed to exist within the domain between the body and control surface. Since the pressure vanishes on the free-surface the integral in (2) may be extended over $S_{bf}$. After a straightforward extension of the vector calculus outlined by Newman (1977, page 133),

$$\begin{aligned} F_i = &\rho \iint_{S_{bf}} \left[ (\phi D_i + \mathbf{S}_i \cdot \nabla\phi)\phi_n - (\tfrac{1}{2}V^2 + gz)(\mathbf{S}_i \cdot \mathbf{n}) \right] dS \\ &- \rho \oint_{C_c} \frac{\partial\varsigma}{\partial t}\phi(\mathbf{S}_i \cdot \mathbf{n}_c) d\ell - \rho \frac{d}{dt} \iint_{S_{bf}} \phi(\mathbf{S}_i \cdot \mathbf{n}) dS \end{aligned} \tag{8}$$

In the line integral, which accounts for the unsteady upper boundary of $S_c$, third-order terms are neglected and the unit normal $\mathbf{n}_c$ is directed away from $S_{bf}$ in the plane $z = 0$. Hereafter we consider only the second-order mean (time-average) components of (8), assuming the first-order motions to be periodic. There is no contribution then from the last term in (8), or from similar total derivatives with respect to time which are neglected in the following equations.

If ($i = 1, 2, 6$) the divergence theorem can be used to replace the integral over $S_{bf}$ on the first line of (8) by an integral over $S_c$. This procedure leads to the well known 'momentum' relations for the mean horizontal drift force and yaw moment.

A more general result follows from (8) by moving the contour $C_c$ in to coincide with the mean position of the body waterline $C_b$, where the normal is $\mathbf{n}_b = -\mathbf{n}_c$. (Here it is assumed for simplicity that the body sides intersect the free surface normally.) Since $\partial\phi/\partial t = -g\varsigma$ to first order, the time derivative of the product $\phi\varsigma$ has zero mean, and

$$-\rho \oint_{C_c} \frac{\partial \varsigma}{\partial t}\phi(\mathbf{S}_i \cdot \mathbf{n})d\ell = \rho \oint_{C_c} \varsigma \frac{\partial \phi}{\partial t}(\mathbf{S}_i \cdot \mathbf{n})d\ell = \rho g \oint_{C_b} \varsigma^2 (\mathbf{S}_i \cdot \mathbf{n})d\ell \tag{9}$$

The contribution from the hydrostatic pressure in (8) is written as the sum of (A) the surface integral over $S_\delta$, defined to be the portion of $S_b$ below the plane $z = 0$ as in (5); (B) the small vertical portion of $S_b$ between $z = 0$ and $z = \varsigma$; and (C) the small horizontal portion of $S_f$ between $C_b$ and its mean position. The contribution from (B) is integrated vertically to give $(-\frac{1}{2})$ of (9). The contribution from (C) is written as a contour integral involving the small width $(\boldsymbol{\xi} \cdot \mathbf{n}_b)$ of the free surface between the exact and mean positions of $C_b$, resulting from horizontal motions of the body. The time derivatives in (C) can be interchanged, as in the first step of (9), and the body boundary condition used to evaluate $\boldsymbol{\xi}_t \cdot \mathbf{n} = \partial\phi/\partial n$. Also in (C), since the normal vector $\mathbf{n}_f$ is vertical on the free surface, $(\mathbf{S}_i \cdot \mathbf{n}_f) = w_i$.

The resulting expression for the generalized mean force is

$$\begin{aligned} F_i = & \rho \iint_{S_0} \left[(\phi D_i + \mathbf{S}_i \cdot \nabla\phi)\phi_n - \tfrac{1}{2}V^2(\mathbf{S}_i \cdot \mathbf{n})\right] dS \\ & + \tfrac{1}{2}\rho g \oint_{C_b} \varsigma^2 (\mathbf{S}_i \cdot \mathbf{n}_b)d\ell - \rho \oint_{C_b} \phi\phi_n w_i d\ell - \rho g \iint_{S_\delta} z(\mathbf{S}_i \cdot \mathbf{n})dS \end{aligned} \tag{10}$$

Since the integrands in the first three integrals are quadratic, in terms of $\phi$, $V$, and $\varsigma$, these integrals can be evaluated over fixed boundaries. The last integral in (10) is over the exact body surface, beneath the plane $z = 0$. Only the second-order components of this integral should be included. In essence, what is required in the last integral is the second-order extension of (5). This is nontrivial, and requires a careful analysis of the mode shapes and their sequence. The attractive feature of (10) is that it isolates these complications in the hydrostatic component, and obviates the need to transfer the first-order hydrodynamic pressure from the moving body surface to its mean position.

For the conventional drift force and moment ($i = 1, 2, ..., 6$) acting upon a body with the same modes of motion, variants of Stokes' theorem can be used to confirm that (10) is equivalent to the pressure-integral results given by Lee & Newman (1991).

## ACKNOWLEDGMENT

This work is supported by the Joint Industry Project 'Wave effects on offshore structures.'

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