RESONANCES OF THE 2-D NEUMANN-KELVIN PROBLEM

J.-M. QUENEZ† AND C. HAZARD†

Introduction.

We are concerned with the linearized 2-dimensional wave resistance problem, i.e., the perturbation of a uniform flow by a fixed rigid body located on the free surface of the fluid. Our purpose is to study how the wave resistance depend on the velocity of the flow, and to determine the local extrema of this function.

A similar question arises for the sea-keeping problem (without forward speed): the determination of the frequencies of the incident wave for which the energy transmitted to the body is maximum. These "resonant states" actually are the traces of complex singularities in the plane of complex frequencies (see e.g. ⟦1⟧). Indeed, the sea-keeping problem can be extended meromorphically to complex values of the frequencies: the poles of this extension are referred to as the "scattering frequencies" of the problem.

Our aim is to show that the same holds for the 2-D Neumann-Kelvin problem and to describe how to extend it to complex values of the flow velocity. Contrary to the sea-keeping problem (or other scattering problems, see ⟦4⟧), this extension cannot be performed by a simple analytic continuation of the Green function: this follows from the anisotropic asymptotic behaviour of the solution at infinity. The method we propose is based on a decomposition of this solution into two parts (see Kuznetsov and Maz'ya ⟦2⟧): a travelling plane wave and the solution of an auxiliary problem of scattering type. The extension of the Neumann-Kelvin problem then amounts to that of the auxiliary problem, for which classical techniques can be used. We show that the resonances of the Neumann-Kelvin problem (i.e. the poles of its extension) are on one hand the scattering frequencies of the auxiliary problem and on the other hand, the zeros of a "passage coefficient" related to this latter problem.

1. The 2-D Neumann-Kelvin problem.

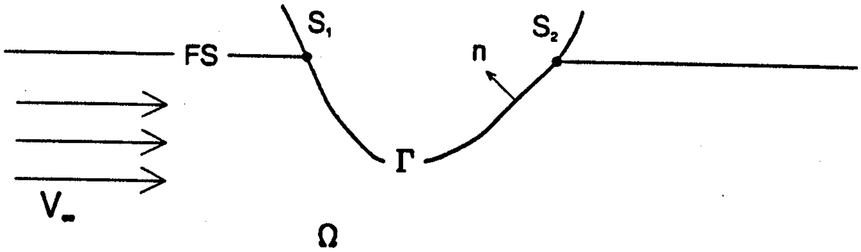
Consider a fixed rigid body placed on the free surface of a uniform flow of a perfect fluid. The system at rest is characterized by the fluid domain Ω and the immersed part $\mathcal{B}$ of the body. The depth is assumed infinite; the boundary $\partial\Omega$ of Ω consists of the free surface FS (located at $z = 0$, where (x, z) are the horizontal and vertical coordinates) and the hull Γ of the body. We denote by n the outwards unit normal on $\partial\Omega$.

The uniform flow is defined by its velocity potential $V_\infty x$ where V_∞ is assumed positive. The perturbation of the flow due to the body is characterized by the perturbed potential φ_1 which satisfies

$$(\mathcal{P}_1) \quad \begin{cases} \text{(a)} \ \Delta\varphi_1 = 0 \ \text{ in } \Omega, \\ \text{(b)} \ \partial_x^2\varphi_1 + \nu\,\partial_n\varphi_1 = 0 \ \text{ on } FS, \\ \text{(c)} \ \lim\limits_{z\to-\infty} \nabla\varphi_1 = 0 \\ \text{(d)} \ \partial_n\varphi_1 = g \ \text{ on } \Gamma, \\ \text{(e)} \ \partial_x\varphi_1(S_2) - \nu \displaystyle\int_\Gamma g\, d\Gamma = \partial_x\varphi_1(S_1) = r, \\ \text{(f)} \ \lim\limits_{x\to-\infty} \nabla\varphi_1 = 0, \end{cases}$$

where ν is proportional to V_∞^2, the function g (defined on Γ) and the real number r are the data (the first equality in (e) shows the conservation of mass flow between upstream and downstream).

† Laboratoire de Simulation et Modélisation des phénomènes de Propagation (URA 853 du CNRS), Ecole Nationale Supérieure de Techniques Avancées, Centre de l'Yvette, Chemin de la Hunière, 91120 Palaiseau, France.



Our purpose is to show that the solution operator associated with problem $(\mathcal{P}_1)$ (i.e., the operator which maps the datum (g,r) onto the solution φ_1) has a meromorphic extension in the complex ν−plane. As mentioned above, this extension leads us to introduce an auxiliary problem $(\mathcal{P}_2)$ of scattering type, which is obtained from $(\mathcal{P}_1)$ by replacing the asymptotic condition (f) by a radiation condition at infinity (in both directions $x \to \pm\infty$) : for a given datum (g,r), it consists in finding a function φ_2 such that

$$(\mathcal{P}_2) \quad \begin{cases} \varphi_2 \text{ satisfies Eqs. (a) to (e)}, \\ \text{(f)} \quad \lim\limits_{R\to\infty} \displaystyle\int_{|x|=R} |\partial_n \nabla\varphi_2 - i\nu\, \nabla\varphi_2|^2\, dz = 0. \end{cases}$$

In the sequel, we will use the following notations: $W^\pm = e^{\nu(z\pm ix)}$ denote two plane waves which propagate, respectively, in the directions $x \to \pm\infty$, and $\Phi^\pm$ are the solutions of problem $(\mathcal{P}_2)$ for the data $(-\partial_n W^\pm, -\partial_x W^\pm(S_1))$ (they represent the scattered potentials associated with the incident waves $W^\pm$). We thus have the following decomposition result (which is proved in § 2 by a slightly different method from [[2]]):

PROPOSITION 1. *Let φ_2 be a solution of the auxiliary problem $(\mathcal{P}_2)$ for the datum (g,r). Then, a solution of $(\mathcal{P}_1)$ for the same datum is given by:*

$$(1) \qquad \varphi_1 = \varphi_2 - \frac{A^-(\varphi_2)}{1+A^-(\Phi^-)}(W^- + \Phi^-) \quad \text{where}$$

$$(2) \qquad A^-(\varphi) = i\int_\Gamma (\varphi\,\partial_n W^+ - \partial_n\varphi\, W^+)\, d\Gamma + \frac{i}{\nu}\left[\varphi\,\partial_x W^+ - \partial_x\varphi\, W^+\right]_{S_1}^{S_2}.$$

In (2), the notation $[\cdots]$ stands for the variation of a function between the two points S_1 and S_2. The physical meaning of $A^-(\varphi)$ is given below.

2. Proof of the decomposition.

First notice that for every complex number T, the function $\varphi_1 = \varphi_2 + T(W^- + \Phi^-)$ satisfies all the equations of $(\mathcal{P}_1)$ save perhaps the condition (f) near $x = -\infty$. The coefficient T must be chosen such that this latter condition is also verified: φ_1 will then be a solution of $(\mathcal{P}_1)$ for the same datum (g,r). In order to estimate the asymptotic behaviour of φ_1, we use an integral representation formula.

Consider the Green functions $G_1(M,P)$ and $G_2(M,P)$ of the Neumann-Kelvin problem $(\mathcal{P}_1)$ and of the auxiliary problem $(\mathcal{P}_2)$. They are solutions of the following equations, respectively for $k = 1,2$:

$$\begin{cases} \text{(a)} \quad \Delta_P G_k(M,P) = \delta_M(P) \;\text{ in } \{z<0\}, \\ \text{(b)} \quad \partial^2_{x_P} G_k(M,P) + \nu\,\partial_{n_P} G_k(M,P) = 0 \;\text{ on } \{z=0\}, \\ \text{(c)} \quad \lim\limits_{z_P\to-\infty} \nabla_P G_k(M,P) = 0, \\ \text{(f)} \quad \begin{cases} \lim\limits_{x_P\to+\infty} \nabla_P G_1(M,P) = 0, \;\text{ if } k=1 \text{ or} \\ \lim\limits_{R\to\infty} \displaystyle\int_{|x_P|=R} |\partial_{n_P}\nabla_P G_2(M,P) - i\nu\nabla_P G_2(M,P)|^2\, dz_P = 0, \;\text{ if } k=2, \end{cases} \end{cases}$$

where $\delta_M(P)$ denotes the Dirac measure at point M. Notice that $G_1(M,P)$ satisfies Eq. (f) near $x_P = +\infty$ whereas the solution φ_1 of $(\mathcal{P}_1)$ satisfies the same condition near $x = -\infty$: this is convenient for the expression of the integral representation formula.

These Green functions are unique up to a constant and are given by

$$(3)\qquad \begin{aligned} G_1(M,P) &= \frac{-1}{2\pi}\log\left(\|PM\|\,\|PM'\|\right) + \frac{1}{2\pi}\mathrm{Pv}\int_{\mathbb{R}} \frac{e^{|\xi|(z_P+z_M)-i(x_P-x_M)}}{|\xi|-\nu}\,d\xi \\ &+e^{\nu(z_P+z_M)}\sin\nu(x_P-x_M), \end{aligned}$$

where $M' = (x_M, -z_M)$, "Pv" denotes the principal value of the integral, and

$$(4)\qquad G_2(M,P) = G_1(M,P) + iW^+(P)W^-(M).$$

The asymptotic behaviour of $G_2(M,P)$ near $x_M = -\infty$ is obtained by noticing that $G_2(M,P)$ is symetric with respect to (M,P) (see [2]):

$$(5)\qquad G_2(M,P) = -\frac{1}{\pi}\log\|OM\| + iW^+(P)W^-(M) + o(\|OM\|^{-1}) \text{ as } x_M \to -\infty.$$

Notice that from Eq. (4), we deduce the expansion of $G_1(M,P)$:

$$G_1(M,P) = -\frac{1}{\pi}\log\|OM\| + o(\|OM\|^{-1}) \text{ as } x_M \to -\infty.$$

The following integral representation formula may be readily proved by classical techniques:

PROPOSITION 2. *For $k = 1,2$, every solution φ_k of problem $(\mathcal{P}_k)$ satisfies:*

$$(6)\qquad \begin{aligned} \varphi_k(M) &= \int_\Gamma \left(\partial_{n_P}G_k(M,P)\varphi_k(P) - G_k(M,P)\partial_n\varphi_k(P)\right)\,d\Gamma(P) \\ &+ \frac{1}{\nu}\left[\partial_{x_P}G_k(M,\cdot)\varphi_k(\cdot) - G_k(M,\cdot)\partial_x\varphi_k(\cdot)\right]_{S_1}^{S_2}. \end{aligned}$$

In order to determine the asymptotic behaviour of $\varphi_2(M)$ near $x_M = -\infty$, we simply have to substitute the expansion (5) of $G_2(M,P)$ near $x_M = -\infty$ in (6) :

$$\varphi_2(M) = A^-(\varphi_2)\; W^-(M) + o(\|OM\|^{-1}) \text{ as } x_M \to -\infty,$$

where $A^-(\varphi_2)$ is defined in (2). This latter quantity is the coefficient of the outgoing wave $W^-(M)$ in the asymptotic behaviour of $\varphi_2(M)$ near $x_M = -\infty$. Remark that the component of the logarithmic term is zero because of the conservation of the mass flow (Eq. (e)). Similarly, we have

$$W^-(M) + \Phi^-(M) = (1 + A^-(\Phi^-))\; W^-(M) + o(\|OM\|^{-1}) \text{ as } x_M \to -\infty.$$

The quantity $(1 + A^-(\Phi^-))$ characterizes the asymptotic behaviour at $-\infty$ of the total potential of the wave W^- perturbed by the obstacle: this is the "passage coefficient" of W^- from $+\infty$ to $-\infty$. Finally,

$$\varphi_1 = \varphi_2 + T(W^- + \Phi^-) = \left(A^-(\varphi_2) + T\left(1 + A^-(\Phi^-)\right)\right) W^-(M) + o(\|OM\|^{-1}) \text{ as } x_M \to -\infty.$$

Since function φ_1 is expected to tend to zero near $-\infty$, this implies formula (1) in proposition 1.

3. Analytic continuation and resonances.

We are now able to prove the meromorphic continuation of the solution operator of the Neumann-Kelvin problem, i.e., the operator $\mathcal{R}_1^{(\nu)}$ which maps the datum (g, r) onto the solution φ_1 of $(\mathcal{P}_1)$. In the same way, we denote by $\mathcal{R}_2^{(\nu)}$ the solution operator of the auxiliary problem $(\mathcal{P}_2)$. According to these notations, formula (1) can be written again in the form:

$$(7) \qquad \mathcal{R}_1^{(\nu)}(g,r) = \mathcal{R}_2^{(\nu)}(g,r) - \frac{A_\nu^-\left(\mathcal{R}_2^{(\nu)}(g,r)\right)}{1 + A_\nu^-\left(\Phi_\nu^-\right)}\left(W_\nu^- + \Phi_\nu^-\right).$$

It is sufficient to prove that each of the two terms in the right-hand side is meromorphic with respect to ν.

On one hand, since $(\mathcal{P}_2)$ is a scattering problem, we know that the operator $\mathcal{R}_2^{(\nu)}$ is meromorphic and that its poles have negative or zero imaginary parts. This may be proved by the same arguments as those used for the sea-keeping problem (see [1],[5]) or Helmholtz equation (see [4]).

On the other hand, the two terms $A_\nu^-\left(\mathcal{R}_2^{(\nu)}(g,r)\right)$ and

$$W_\nu^- + \Phi_\nu^- = W_\nu^- + \mathcal{R}_2^{(\nu)}(-\partial_n W_\nu^-, -\partial_x W_\nu^-)$$

are clearly meromorphic (because of the properties of $\mathcal{R}_2^{(\nu)}$) and admits at most the same poles as $\mathcal{R}_2^{(\nu)}$. The last term to be studied is the inverse of the passage coefficient $1 + A_\nu^-\left(\Phi_\nu^-\right)$. This coefficient is meromorphic so that its inverse is meromorphic. It is clear that the poles of the inverse are the zeros of this coefficient. When this occurs, that is when $1 + A_\nu^-\left(\Phi_\nu^-\right) = 0$, the function $W_\nu^- + \Phi_\nu^-$ is a solution of $(\mathcal{P}_1)$ with homogeneous datum $(0,0)$.

We thus have proved:

THEOREM. *The solution operator $\mathcal{R}_1^{(\nu)}$ associated with $(\mathcal{P}_1)$ is a meromorphic operator valued function of ν. Its poles are*

- *either the poles of $\mathcal{R}_2^{(\nu)}$, the meromorphic continuation of the solution operator $\mathcal{R}_2^{(\nu)}$ of $(\mathcal{P}_2)$ to complex values of ν,*
- *or the zeros of $1 + A_\nu^-\left(\Phi_\nu^-\right)$, the passage coefficient of $(\mathcal{P}_2)$ for the plane wave W_ν^-.*

References.

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