On uniqueness in linearized two-dimensional water-wave problem for two surface-piercing bodies

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Introduction

In the study of linearized water waves interacting with obstacles, the question of the uniqueness of the solution is not yet fully answered. That is, are there non-radiating (and therefore persistent) oscillatory modes at any frequency for some geometry? John (1950) established uniqueness for the case where the body is surface-piercing and has the property that vertical lines from the free surface do not intersect the body. More recently, Simon and Ursell (1984) generalized John's approach to prove uniqueness for a wider class of problem. Each of these papers uses a bound on the potential energy of the non-radiating motion relative to its kinetic energy; as these quantities are equal a contradiction is established. However, this approach cannot be employed directly in two dimensions when there are two surface piercing bodies, essentially because the free surface between the bodies is separated (by the bodies) from both $\pm\infty$. The purpose of the present work is to show how a conformal mapping can be used to help to derive a bound on this part of the potential energy, and thereby prove uniqueness; the end result is that the solution will be unique provided the frequency is no greater than a parameter which depends on the geometry.

Statement of the problem

We consider the small-amplitude two-dimensional irrotational motion of an ideal fluid; the motion is assumed periodic with frequency ω, and so is described by a velocity potential $\mathrm{Re}\{u(x,y)e^{-i\omega t}\}$, where (x,y) are Cartesian coordinates with origin in the mean free surface and y measured vertically upwards.

The complex function u must satisfy the following boundary-value problem:

$$\nabla^2 u = 0 \quad \text{in} \quad W, \tag{1}$$

$$u_y - \nu u = 0 \quad \text{on} \quad F, \tag{2}$$

$$\partial u/\partial n = f \quad \text{on} \quad S, \tag{3}$$

$$\partial u/\partial|x| - i\nu u = o(1) \quad \text{as} \quad |x| \to \infty. \tag{4}$$

Here W denotes the region occupied by fluid which is assumed to be of infinite depth, and S is the union of the wetted surfaces of all the bodies (submerged totally or partially). F is the free surface which is the part of $y = 0$ outside all bodies.

The parameter ν is positive as it is equal to $\omega^2/g; g$ is the acceleration of gravity. We also assume that $u \in L_\infty(W) \cap H^1(W_0)$ for any bounded region $W_0 \subset W$, where H^1 denotes a Sobolev space.

We now suppose that there are *two* surface-piercing bodies; we shall prove that the homogeneous ($f \equiv 0$) problem has only a trivial solution, and hence that the inhomogeneous problem has a unique solution, for a range of values of ν.

Let the two bodies occupy domains D_+ and D_-, and let them be contained inside semicircles of radius r_+ and r_- respectively, each centred on the mean free surface. For the proof that follows we need to assume that each body is in contact with its circumscribing semicircle at the waterline position nearest the other body; thus the distance ℓ between the semicircles is also the distance between the bodies. Label the other waterline intersections by p_+ and p_- respectively.

We can choose the origin in the x-axis so that these semicircles will coincide with coordinate σ-lines of the bipolar system (σ, τ) :

$$x = \frac{a \sinh \tau}{\cosh \tau - \cos \sigma}, \quad y = \frac{a \sin \sigma}{\cosh \tau - \cos \sigma}. \tag{5}$$

For this purpose we have to find (see e.g. Morse & Feshbach, 1953) positive constants $b_\pm, d_\pm, a$ such that

$$\begin{aligned} a = r_- \sinh d_- &= r_+ \sinh d_+, & b_+ + b_- &= \ell \\ b_- + r_- &= r_- \cosh d_-, & b_+ + r_+ &= r_+ \cosh d_+. \end{aligned} \tag{6}$$

One can easily verify that this system of equations has the unique solution such that

$$\cosh d_\pm - 1 = \frac{\ell^2 + 2\ell r_\mp}{2r_\pm(r_+ + r_- + \ell)}. \tag{7}$$

We can assume $r_- \geq r_+$ without loss of generality, and so $d_- \leq d_+$. We shall prove the following

Theorem Let the set $D_- \cup D_+$ be enclosed between two rays from the points p_- and p_+ at an angle $\pi/2 - \beta$ to the vertical. Let also $D_\pm \subset \{[x \mp (b_\pm + r_\pm)]^2 + y^2 < r_\pm^2, y < 0\}$.

If the inequality

$$2 - \operatorname{cosec}^2\beta > \frac{\ell^2 + 2\ell r_-}{2r_+(r_+ + r_- + \ell)} \tag{8}$$

holds, then the homogeneous ($f \equiv 0$) problem (1) - (4) has only trivial solutions for all positive values of the parameter ν that do not exceed

$$\frac{2(2 - \operatorname{cosec}^2\beta)r_+(r_+ + r_- + \ell) - (\ell^2 + 2\ell r_-)}{2\pi r_+[(\ell^2 + 2\ell r_-)(\ell^2 + 2\ell r_- + 4r_+\{r_+ + r_- + \ell\})]^{\frac{1}{2}}}. \tag{9}$$

Proof

Let F_- be the free surface between the bodies, and F_+ that outside.

The mapping $x + iy \longrightarrow \sigma + i\tau$ (see (5)) conformally transforms the lower half-plane into the strip $\{-\pi < \sigma < 0, -\infty < \tau < \infty\}$. By the hypothesis of the theorem the rectangle $R = \{-\pi < \sigma < 0, \ -d_- < \tau < +d_+\}$ is contained in the image of W. Furthermore, the side $\{\sigma = -\pi, -d_- < \tau < +d_+\}$ is the image of F_- and the side $\{\sigma = 0, \ -d_- < \tau < +d_+\}$ is a subset of the image of F_+.

From the boundary-value problem we get, with Green's formula, the identity

$$\int_W |\nabla u|^2 dxdy = \nu \int_{F_- \cup F_+} |u|^2 dx. \tag{10}$$

It is straightforward to show that

$$\nu \int_{F_+} |u|^2 dx \leq \frac{1}{2} \operatorname{cosec}^2\beta \int_W |\nabla u|^2 dxdy \tag{11}$$

and that

$$\nu \int_{F_-} |u|^2 dx \leq \frac{1}{2} \nu a \int_{-d}^{+d_+} |v(-\pi, \tau)|^2 d\tau$$

where $v(\sigma, \tau) = u(x, y)$. Also, we show that

$$\begin{aligned} \textstyle\int_{-d_-}^{+d_+} |v(-\pi,\tau)|^2 d\tau &\leq 2\left\{\textstyle\int_{-d_-}^{+d_+} |v(0,\tau)|^2 d\tau + \pi \int_R |v_\sigma(\sigma,\tau)|^2 d\sigma d\tau\right\} \\ &\leq 2\left\{\textstyle\int_{-d_-}^{+d_+} |v(0,\tau)|^2 d\tau + \pi \int_W |\nabla u|^2 dxdy\right\}, \end{aligned} \tag{12}$$

and finally, that

$$\int_{-d_-}^{+d_+} |v(0,\tau)|^2 d\tau \leq (\cosh d_+ - 1) \int_{F_+^e} |u|^2 dx \leq \frac{1}{2}(\cosh d_+ - 1) \int_{W^e} |\nabla u|^2 dxdy \tag{13}$$

where F_+^e and W^e are the parts of F_+ and W external to the semicircles bounding $D_\pm$.

Equation (10)-(13) together give

$$\int_W |\nabla u|^2 dxdy \leq \left\{\frac{1}{2} \operatorname{cosec}^2\beta + A + \pi B\nu\right\} \int_W |\nabla u|^2 dxdy \tag{14}$$

where $A = \frac{1}{2}(\cosh d_+ - 1) = \dfrac{\ell^2 + 2\ell r_-}{4r_+(r_+ + r_- + \ell)}$ and $B = a = \{R(R + 2r_+)\}^{1/2}$ where $R = r_+(\cosh d_+ - 1) = 2r_+A$. If $A + \beta\nu\pi < 1 - \frac{1}{2}\operatorname{cosec}^2\beta$, there will be a contradiction in (14) unless $\nabla u \equiv 0$ (and so $u \equiv 0$ since $u \to 0$ as $x \to \pm\infty$); this proves uniqueness provided ν does not exceed the value (9).

Discussion

This paper has described the uniqueness proof for a class of problems involving two surface-piercing bodies. One limit of this is to let r_- tend to infinity, keeping ℓ and r_+ fixed; the resultant geometry involves one body in deep water in the presence of a cliff (which may have an overhang). A modification of the method of this paper also allows the corresponding finite-depth problem to be studied, and an upper bound on ν found for uniqueness. Further modification allows the problem of two or three bodies in a uniform depth ocean to be studied, in each case yielding an upper bound on the frequency. It is hoped other conformal transformation may lead to other uniqueness proofs.

References

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