Note on the Cauchy-Poisson problem for finite depth in three dimensions

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1 Introduction

The Cauchy-Poisson problem treats the waves generated by an instantaneous localised splash. Waves spread out from the disturbance, the long waves travelling faster than the short waves. We are concerned with the behaviour near the wave front $r = t(gh)^{1/2}$ for large t. The corresponding velocity potential is readily expressed as an integral which we may expect to be able to treat by some modification of the Method of Steepest Descents. This problem has been much discussed in recent years, with particularly interesting contributions from J.N.Newman who has found a leading term involving the square of an Airy function whereas the leading term in two dimensions involves just the Airy function itself. Newman's result is valid in a narrow zone near the wave front. During the past few years I have tried to find a more complete expression but my attempt has met with only limited success. My asymptotic expansion was found to involve modified Airy integrals like

$$\int \frac{\exp(\frac{1}{3}iu^3 + izu)}{u^{1/2}} du, \tag{1}$$

and I have found that this does not imply a contradiction with Newman's result because it can be shown that this integral (with suitable limits of integration) is in fact the square of an Airy function. My asymptotic expansion, like Newman's, was found to be valid only in a narrow zone near the wave front. The reason is, that in three dimensions the integrand has three nearly coincident critical points, (two saddle points and a branch point,) whereas in two dimensions the integrand has only two nearly coincident critical points (two saddle points). It may therefore be expected that the asymptotic expansion in three dimensions depends on two variables rather than one variable as in two dimensions. Here I wish to describe a new approach which promises to give a wider zone of validity.

2 The two-dimensional problem

The two-dimensional problem for an initial surface impulse leads to a consideration of the integral

$$\phi(x, 0, t) = \int_0^\infty A(k) \cos kx \, \cos \sigma t \, dk, \tag{2}$$

where $\sigma^2 = gk \tanh kh$, and where $A(k)$ is a coefficient function depending on the shape of the initial impulse. The phase involves terms like

$$\begin{aligned}(-ikx + i\sigma t) &= iT[(u \tanh u)^{1/2} - (X/T)u] \quad (3)\\ &= iT[(T-X)u/T - u^3/6 + \ldots], \quad (4)\end{aligned}$$

where $u = kh, T = t(g/h)^{1/2}, X = x/h$, and T/X is close to 1 near the wave front. This phase has two stationary points near $u = 0$. If we neglect higher powers of u in the expansion, we obtain an Airy function

$$Ai(Z) = \frac{1}{\pi}\int_0^\infty \cos(\frac{1}{3}w^3 + Zw)dw \quad (5)$$

of the variable $Z = 2^{1/3}T^{2/3}((X/T) - 1)$, but because of the neglect of higher powers of u this approximation is valid only in a narrow zone of T/X near 1, and does not readily join up with the stationary-phase expansions away from the wave front. However, the region of validity can be extended to a wide zone by the following method. Introduce a new variable of integration v by means of the transformation

$$(u \tanh u)^{1/2} - (X/T)u = \epsilon v - v^3/6 \text{ exactly;} \quad (6)$$

it can be shown that such a transformation exists. The parameter $\epsilon(X/T)$ is determined from the condition that the points of stationary phase on the left and on the right must correspond. We then obtain an Airy function of a slightly different variable $2^{1/3}T^{2/3}\epsilon(X/T)$, and the resulting expansion is found to be valid in a wide zone including the wave front.

3 The three-dimensional problem

Let us try to apply a similar method to the three-dimensional Cauchy-Poisson problem. The three-dimensional problem for an initial surface impulse leads to a consideration of the integral

$$\phi(r,0,t) = \int_0^\infty A(k)\, J_0(kr) \cos \sigma t\, kdk, \quad (7)$$

where $\sigma^2 = gk \tanh kh$, where $J_0(Z)$ is the usual Bessel function, and where $A(k)$ is a coefficient function depending on the shape of the initial impulse. If now we use the asymptotic expansion

$$J_0(Z) \sim (2/\pi Z)^{1/2} \cos(Z - \pi/4), \quad (8)$$

valid for large Z, we obtain an integral involving circular functions, to which the Method of Stationary Phase may be applied. For an observer near the wave front the phase has two nearly coincident stationary points near $k = 0$, as we would expect, and for the potential we thus obtain a result which involves integrals like (1). We now observe a difficulty: we have used the asymptotic expansion (8) for $J_0(kr)$ which is valid for large kr, but near the points of stationary phase the wavenumber k is small, and although r is large it is not obvious that the product kr can be treated as large. It can nevertheless be shown by a lengthy calculation that the result is correct, but only in a small region near the wave front.

In my recent work I have tried to use the Method of Stationary Phase for double integrals. The integral in (7) can be written as the double integral

$$\int_{-\pi}^{\pi} d\alpha \int_0^{\infty} k dk A(k) \exp(-ikr\cos\alpha)\cos\sigma t, \tag{9}$$

which involves only circular functions, so that Stationary Phase is applicable. There are 4 points of stationary phase (2 real and 2 pure imaginary) near $(\alpha = 0, k = 0)$ in the four-dimensional space of the two complex variables α and k. Near $(\alpha = 0, k = 0)$ the exponent of the exponential $\exp(-ikr\cos\alpha + i\sigma t)$ can be expanded in powers of α and k:

$$\begin{aligned}(-ikr\cos\alpha + i\sigma t) &= iT[(u\tanh u)^{1/2} - (R/T)u\cos\alpha] &(10)\\ &= iT[(T-R)u/T - u^3/6 + (R/2T)u\alpha^2 + \ldots], &(11)\end{aligned}$$

where $u = kh, T = t(g/h)^{1/2}, R = r/h$, and T/R is close to 1 near the wave front. We may obtain an asymptotic expansion by retaining only these cubic terms, but because of the neglect of higher terms we can again expect only a narrow region of validity. To obtain a wider region I am trying a transformation of variables of integration from the variables (u, α) to new variables (v, β), where

$$(u\tanh u)^{1/2} - (R/T)\cos\alpha = \epsilon v - v^3/6 + (R/2T)v\beta^2 \quad \text{exactly,} \tag{12}$$

where $\epsilon(R/T)$ is the same small parameter as in two dimensions, with R/T in place of X/T; it can be shown that such a transformation exists. This is analogous to the transformation which gives the wide region of validity in the two-dimensional Cauchy-Poisson problem. We find that we obtain 4 nearly coincident saddle points near the origin $(v = 0, \beta = 0)$, and we note that the origin lies on the boundary of the region of integration. We again obtain integrals like (1) but involving the slightly different variable $\epsilon(R/T)$, and higher approximations can be found by a systematic procedure. This work is still in progress.