The reciprocal relation of hydrodynamic forces acting on floating bodies in both waves and slow current

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Ocean structures are often working in an environment of both waves and current. Usually, the velocity of current, i.e. U, is quite small compared to the phase velocity of the incident waves. The amplitude of the incident wave is assumed to be small compared to the wave length. Therefore terms of order $O(U^2)$(or higher) and nonlinear in wave amplitude are dropped in the present work. The viscous effect is also neglected. The fluid is incompressible and the flow is irrotational so that there exists a velocity potential. After establishing the boundary value problem for the velocity potential, we focus on the calculation of hydrodynamic forces and verification of reciprocal relations between these terms . It turns out that the wave exciting forces to the order of $O(U)$ can be evaluated by the potentials without curren, which is an extension of the results obtained by Wu & Eatock-Taylor (1990) for added mass and damping coefficients.

1. Formulation of the problem

The problem to be considered is that some bodies are floating in a water of depth h. In addition to a uniform current with velocity U, there comes a plane wave with a frequency of ω_0. The bodies are restrained from the drift motions but are free to linear oscillation at encountering frequency ω. As an extension of the previous work(Bao & Kinoshita, 1994) in which only diffraction problem was considered, the radiation problem will be included in the present work. A right-handed coordinate system is adopted. The plane $z=0$ coincides with the still water free surface and z-axis is positive upwards. The x-axis points against the uniform flow so that the current is moving in the negative x direction.

The potential Φ_T is decomposed into a uniform flow, a steady disturbance potential $\bar{\phi}$ and an unsteady potential. The unsteady potential is in turn separated into diffraction potential and radiation potentials, i. e.

$$\Phi_T(\mathbf{x},t) = -Ux+U\bar{\phi}(\mathbf{x})+Re\left\{\left[-i\omega\sum_{j=1}^{6}\xi_j\phi_j(\mathbf{x})+\frac{\zeta_0 g}{i\omega_0}\phi_D(\mathbf{x})\right]e^{-i\omega t}\right\} \tag{1}$$

where the encountering frequency is given by $\omega = \omega_0 - Uk_0\cos\beta$ with β the incident wave angle referring to the positive x-axis; ϕ_j $(j = 1 - 6)$ represents the radiation potentials corresponding to six modes of oscillation motion respectively and ϕ_D indicates the scattering potential which includes an incident wave potential ϕ_0 and a diffraction potential ϕ_7.

Based on the assumption of small current velocity, terms of order $O(U^2)$ are dropped. The steady disturbance potential satisfies a rigid wall condition on the free surface and can be solved by double-model method.

In addition to the governing Laplace equation and rigid wall condition on the sea bottom, each mode of the unsteady potentials satisfies the following boundary condition on the free surface and body surface respectively.

$$\frac{\partial \phi_j}{\partial z} - \nu\phi_j - 2i\tau W\cdot\nabla\phi_j + i\tau\phi_j \frac{\partial^2 \bar{\phi}}{\partial z^2} = O(\tau^2) \qquad \text{at } z=0 \quad (j=1\text{-}7) \qquad (2a)$$

$$\frac{\partial \phi_j}{\partial n} = \begin{cases} n_j - \tau m_j/i\nu & (j=1-6) \\ -\partial\phi_0/\partial n & (j=7) \end{cases} \qquad \text{at body surface } S_0 \qquad (2b)$$

where

$$\tau = \omega U/g \qquad (3a)$$

$$\nu = \omega^2/g \qquad (3b)$$

$$W = \nabla(-x + \bar{\phi}) \qquad (3c)$$

$(n_1, n_2, n_3) = \mathbf{n}$, the inward unit normal vector of the body (3d)

$$(n_4, n_5, n_6) = \mathbf{x}\times\mathbf{n} \qquad (3e)$$

$$(m_1, m_2, m_3) = -(\mathbf{n}\cdot\nabla)\, W; \qquad (3f)$$

$$(m_4, m_5, m_6) = -(\mathbf{n}\cdot\nabla)(\mathbf{x}\times W); \qquad (3g)$$

Then each mode of unsteady potentials is expanded into power series of τ, i. e.

$$\phi_j = \phi_j^{(0)} + \tau\phi_j^{(1)} + O(\tau^2) \qquad (4)$$

Substituting this expansion into the above boundary conditions and collecting terms with the same power of τ, we can obtain the boundary conditions satisfied by potentials with different orders. For the zeroth order, it is the same as those in the problem without current:

$$\frac{\partial \phi_j^{(0)}}{\partial z} - \nu\phi_j^{(0)} = 0 \qquad \text{at } z=0 \quad (j=1\text{-}7) \qquad (5a)$$

$$\frac{\partial \phi_j}{\partial n} = \begin{cases} n_j & (j=1-6) \\ -\partial\phi_0/\partial n & (j=7) \end{cases} \qquad \text{at body surface } S_0 \qquad (5b)$$

The current effects including the contribution from the steady disturbance potential $\bar{\phi}$ enter in the first order problem, i. e.

$$\frac{\partial \phi_j^{(1)}}{\partial z} - \nu\phi_j^{(1)} = f_j \qquad \text{at } z=0 \quad (j=1\text{-}7) \qquad (6a)$$

$$\frac{\partial \phi_j^{(1)}}{\partial n} = \begin{cases} -\dfrac{m_j}{i\nu} & (j=1-6) \\ 0 & (j=D) \end{cases} \qquad \text{at } S_0 \qquad (6b)$$

where $$f_j = 2i\, W\cdot\nabla\phi_j^{(0)} - i\phi_j^{(0)}\frac{\partial^2\bar{\phi}}{\partial z^2} + \delta_{7j}(2i\,\nabla\bar{\phi}\cdot\nabla\phi_0 - i\phi_o\frac{\partial^2\bar{\phi}}{\partial z^2}) \qquad (6c)$$

According to Bao & Kinoshita (1994), the far field condition for each order of potentials is given by

$$\lim_{r\to\infty}\sqrt{r}\,(\frac{\partial \phi_j^{(0)}}{\partial r} - ik\phi_j^{(0)}) = 0 \qquad \text{at } r\to\infty \qquad (7a)$$

$$\frac{\partial \phi_j^{(1)}}{\partial r} - ik\phi_j^{(1)} - ik_1\phi_j^{(0)} \rightarrow 0 \qquad \text{at } r \rightarrow \infty \tag{7b}$$

where $k_1 = \dfrac{2k\cos\theta}{\tanh kh + kh\,\mathrm{sech}^2 kh}$ (7c)

and $k \tanh kh = \nu$ (7d)

Because of the perturbation method used in the present work, the far field condition implies that $\phi_j^{(1)}$ would be divergent at far field. Therefore, the solution is regarded as a "near field" one which is valid only in the region of $r < \infty$.

2. Hydrodynamic Forces

The problem can be solved by the eigenfunction expansion method in a similar way to solving the diffraction problem in our previous work(Bao & Kinoshita, 1994). To apply the method to the interaction problem of multiple bodies, it just needs to replace the diffraction of incident wave by the radiated waves for each body.

Once the potentials are solved, the hydrodynamic forces can be evaluated by integrating the hydrodynamic pressure over the wetted surface of the bodies. The hydrodynamic coefficients, i.e. the added mass and damping coefficients, are given by

$$F_{ij} = \omega\mu_{ij} - \lambda_{ij} = \rho\int_{S_0}\left[i\omega\phi_j - UW\cdot\nabla\phi_j\right] n_i ds \tag{8}$$

By the Tuck theorem(Ogilvie & Tuck, 1969), the integral involving $\underline{W}$ can be evaluated by

$$\int_{S_0}\left(W\cdot\nabla\phi_j\right) n_i ds = -\int_{S_0}\phi_j m_i ds + \int_{C_0}\phi_j\frac{\partial\bar{\phi}}{\partial z} n_i di \tag{9}$$

where C_0 is the waterline of the body. Because of the rigid wall condition satisfied by the steady disturbance potential $\bar{\phi}$, the integral along waterline C_0 vanishes. Thus the hydrodynamic coefficients may be written as

$$F_{ij} = \rho i\omega\int_{S_0}\phi_j^{(0)} n_i ds + \tau\rho i\omega\int_{S_0}\left(\phi_j^{(1)} n_i + \frac{m_i}{i\nu}\phi_j^{(0)}\right) ds + O(\tau^2) \tag{10}$$

The integral of the first order potential can be transferred to the integral involving the zeroth order potential only by using the Green's theorem.

$$\int_{S_0}\phi_j^{(1)} n_i ds = \int_{S_0}\phi_i^{(0)}\frac{\partial\phi_j^{(1)}}{\partial n} ds - \int_{S_F+S_\infty}\left[\phi_j^{(1)}\frac{\partial\phi_i^{(0)}}{\partial n} - \phi_i^{(0)}\frac{\partial\phi_j^{(1)}}{\partial n}\right] ds \tag{11}$$

By means of the boundary conditions satisfied by different orders of potentials on the free surface and body surface as well as at far field, it can be shown that

$$F_{ij} = i\omega\mu_{ij} - \lambda_{ij} = i\omega a_{ij} - b_{ij} + \tau\left\{\rho\frac{\omega}{\nu}\int_{S_0}\left(m_i\phi_j^{(0)} - m_j\phi_i^{(0)}\right) ds - \rho\omega\int_{S_F}\left[\phi_i^{(0)}\left(\mathbf{W}\cdot\nabla\phi_j^{(0)}\right) - \phi_j^{(0)}\left(\mathbf{W}\cdot\nabla\phi_i^{(0)}\right)\right] ds\right\} \tag{12}$$

where a_{ij} and b_{ij} is added mass and damping coefficient without current respectively, i.e. the first term on the right hand side of (10). At this stage, it is easily to verify the well known Timman-Newman relation(Timman & Newman, 1962), i. e.

$$F_{ij}(U) = F_{ji}(-U) \tag{13}$$

where $F_{ji}(-U)$ indicates the result of the inverse flow problem. It is readily obtained by replacing τ in the above formulae by $-\tau$. An immediate corollary is that

$$F_{ii}(U) = F_{ii}(0) + O(\tau^2) \tag{14}$$

This means that up to the order of τ the diagonal added mass and damping coefficients depend on the current speed only through encountering frequency.

When the wave exciting forces are considered, a similar deduction can be made as follows,

$$F_i = \frac{\rho g \zeta_0}{i\omega_0}\int_{S_0}\left[i\omega\phi_D - U\mathbf{W}\cdot\nabla\phi_D\right] n_i ds$$

$$= \frac{\rho g \zeta_0 \omega}{\omega_0}\int_{S_0}\phi_D^{(0)} n_i ds + \tau\frac{\rho g \zeta_0 \omega}{\omega_0}\int_{S_0}\left(\phi_D^{(1)} n_i + \frac{m_i}{i\nu}\phi_D^{(0)}\right) ds + O(\tau^2) \tag{15}$$

Noticing that $\phi_D^{(1)}$ satisfies a homogeneous boundary condition on the body surface S_0 the final result is given by

$$F_i = F_i^{(0)} + \tau\rho g\zeta_0\left\{\frac{1}{\nu}\int_{S_0}(m_i - n_i k_0 \cos\beta)\phi_D^{(0)} ds\right.$$

$$+ i\int_{S_F}\phi_i^{(0)}\left(2\nabla\bar{\phi}\cdot\nabla\phi_0 - \phi_0\frac{\partial^2\bar{\phi}}{\partial z^2}\right) ds$$

$$\left.+\int_{S_F}\left[\phi_i^{(0)}\left(\mathbf{W}\cdot\nabla\phi_7^{(0)}\right) - i\phi_7^{(0)}\left(\mathbf{W}\cdot\nabla\phi_i^{(0)}\right)\right] ds\right\} + O(\tau^2) \tag{16}$$

where $F_i^{(0)}$ is the wave exciting force without current effect given by the first integral in the last line of (15).

Once the above potentials and motion amplitudes are solved the wave drift force and wave drift damping can also be calculated. Some numerical examples will be presented later.

Reference

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3) Ogilvie, T.F. & Tuck, E.O., 1969, A rational strip theory for ship motions: part 1. *Dept. Nav. Arch. Mar. Eng. Rep. No. 013*, Univ. of Michigan.

4) Timman, R. & Newman, J.N., 1962, The coupled damping coefficients of a symmetric ship, *J. Ship Res.* vol. 5, 1-7.

5) Wu, G.X. and Eatock-Taylor, R.,1990, The hydrodynamic forces on an oscillating ship with low forward speed, *J. Fluid Mech.*, 211,333-353.

DISCUSSION

Eatock Taylor R.: Have you any comments on the convergence of the free surface integrals in your expressions for wave excitation and hydrodynamic forces, for the cases you are considering?

Bao & Kinoshita: It is a sacrifice that we have to do integral over free surface if we try to evaluate hydrodynamic forces without solving the first order problem. Terms associated with undisturbed uniform flow in the integral do cause some trouble in convergence. In the case of circular cylinder, we found that these terms will converge at an order of $1/r^2$. So the integral over free surface exists. In numerical calculation, we have to extend the integral range to the place 20 times of the cylinder radius away from the cylinder to get enough accuracy.

Grue J.: How does the method converge when evaluating wave drift damping of a freely floating body? What is the truncation radius at the free surface?

Bao & Kinoshita: The wave drift damping can only be evaluated by integrating the pressure over the wetted surface of cylinders after solving for each order of potential in our method. This integral will not cause any problems of convergence. However, to seek a particular solution which satisfies the inhomogeneous free surface condition for the first order potential, it will be involved in an integral over the free surface. Since the particular solution corresponding to the inhomogeneous terms associated with uniform flow, which will indeed cause the integral divergent, has been obtained by a derivative operator, the remaining terms are all associated with steady disturbance potential which decays fast enough to ensure the convergence of the integral. Therfore, the truncation radius of this integral over free surface could be the same as used in the boundary element method.