The Role of Irregular Frequencies in the Transient Neumann-Kelvin Problem *

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1 Introduction

In this abstract we show that the commonly used transient integral equation representations of the solution to the Neumann-Kelvin problem, when derived using the free-surface Green function, have solutions that exist and are unique. In contrast to the corresponding integral equations in the frequency domain, neither a distribution of transient sources nor one of both sources and normal dipoles admits a non-trivial homogeneous solution to the integral equation. Transient solutions to the continuous integral equation are therefore free of the effects of what are usually called the irregular frequencies. Discrete numerical solutions to the integral equations however, because of the error introduced through discretization in both space and time, behave in a way that is qualitatively similar to their frequency-domain counterparts.

2 The Transient Problem

We will discuss the Neumann-Kelvin problem for a body with zero forward speed for simplicity, although the analysis of the transient integral equation is essentially identical at non-zero forward speed. Subject to the assumptions of a potential flow, we consider a perturbation velocity potential $\phi(\vec{x},t)$ that satisfies the following linearized initial-boundary-value problem:

$$\begin{aligned} \nabla^2\phi &= 0 && \text{in } \mathcal{V} \\ \phi_{tt}+g\phi_z &= 0 && \text{on } z=0 \\ \vec{n}\cdot\nabla\phi &= V_n && \text{on } \bar{S}_b \\ \phi=\phi_t &= 0 && \text{on } z=0,\ \text{at } t=0, \end{aligned} \tag{1}$$

where $\mathcal{V}$ is the fluid exterior to the body with its surface defined by $\bar{S}_b$, g is the acceleration due to gravity, and $V_n(\vec{x},t)$ is the component of the linearized velocity of a point on the body surface in the direction of $\vec{n}$, the normal vector to $\bar{S}_b$ directed out of the fluid. [Subscripts $\vec{x},t$ denote partial differentiation with respect to the subscript variable.] The coordinate system is right-handed with z positive upwards and $z=0$ the plane of the quiescent free-surface. This problem is well-posed and a unique solution exists.

A suitable Green function for this problem is:

$$G(\vec{x};\vec{\xi},t)=G^{(0)}(\vec{x};\vec{\xi})+H(\vec{x};\vec{\xi},t), \tag{2}$$

where

$$G^{(0)}=\left(\frac{1}{r}-\frac{1}{r'}\right),\ H=2\int_0^\infty dk\,[1-\cos(\sqrt{gk}\,t)]\,e^{kZ}\,J_0(kR),$$

$$\left.\begin{matrix} r \\ r' \end{matrix}\right\}=\sqrt{(x-\xi)^2+(y-\eta)^2+(z\mp\zeta)^2},\quad Z=(z+\zeta),\quad R=\sqrt{(x-\xi)^2+(y-\eta)^2},$$

and J_0 is the Bessel function of order zero.

*This work is supported by the Office of Naval Research, contract #N00014-90-J1160.

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An integral equation representation of the initial-boundary-value problem may be derived by applying Green's theorem to the the Green function $G(\vec{x};\vec{\xi},t-\tau)$, and the time derivative of the potential $\phi_\tau(\xi,\tau)$, and integrating in time:

$$2\pi\,\phi(\vec{x},t) \; + \int\!\!\int_{S_b} d\vec{\xi}\,\phi(\vec{\xi},t)\,G^{(0)}_{n_\xi}(\vec{x},\vec{\xi}) + \int_0^t d\tau \int\!\!\int_{S_b} d\vec{\xi}\,\phi(\vec{\xi},\tau)\,G_{tn_\xi}(\vec{x},\vec{\xi},t-\tau) \tag{3}$$

$$= \int\!\!\int_{S_b} d\vec{\xi}\,G^{(0)}(\vec{x},\vec{\xi})\,\phi_n(\vec{\xi},t) + \int_0^t d\tau \int\!\!\int_{S_b} d\vec{\xi}\,G_t(\vec{x},\vec{\xi},t-\tau)\,\phi_n(\vec{\xi},\tau).$$

This is a Fredholm second-kind equation in the spatial variables, and appears to be a Volterra second-kind equation in time. Because of the time-symmetry properties of the kernal however, it is more properly described as simply a Fredholm second-kind equation with time as a parameter and a forcing which depends upon the solution over the entire time history. More formally, consider the time integral on the left-hand side of Equation (3) in terms of the definition of a definite integral:

$$\int_0^t d\tau \int\!\!\int_{S_b} d\vec{\xi}\,\phi(\vec{\xi},\tau)\,G_{tn_\xi}(\vec{x},\vec{\xi},t-\tau) = \lim_{\substack{M\to\infty \\ \max \Delta_n t \to 0}} \sum_{n=1}^{M} h(t_n^*)\,\Delta_n t.$$

Here $t_n(n = 0,1,...,M)$ is a sequence of times over the interval such that $0 = t_0 < t_1 < ... < t_M = t$, $\Delta_n t = t_n - t_{n-1}$, and the value t_n^* is any time satisfying $t_{n-1} \le t_n^* \le t_n$. Since the function $h(t)$ is a continuous function of time this definition is unambiguous (except possibly at $t = 0$). The last interval of this summation does not contribute to the integral because as $\Delta t_M \to 0$, $t_M^* \to t$; and $h(t_M) \to 0$ since $G_t(\vec{x};\vec{\xi},0) = G_{tn_\xi}(\vec{x};\vec{\xi},0) = 0$ (the symmetry property). Thus the integral over the closed interval $[0,t]$ is equivalent to the integral over the semi-open interval $[0,t)$ in which the unknown potential $\phi(\vec{x},t)$ does not appear. This allows Equation (3) to be written

$$2\pi\,\phi(\vec{x},t) + \int\!\!\int_{S_b} d\vec{\xi}\,\phi(\vec{\xi},t)\,G^{(0)}_{n_\xi}(\vec{x},\vec{\xi}) = f(\vec{x},t), \tag{4}$$

where $f(\vec{x},t)$ is known provided that the solution has been advanced to time t from the initial conditions. Equation (4) is the integral equation representation of the exterior Neumann problem, with $\phi(\vec{x},t)$ anti-symmetric about $z = 0$, and it has a unique solution [3]. Daoud [2] reached the same conclusion, in a slightly different context and in two dimensions, using the physical argument that between the time $t - \Delta t$ and the time t, gravity has not had time to take effect.

A source-only integral equation for this problem can be derived in the usual way by defining a potential $\phi'(\vec{x},t)$ due to a fictitious flow in the interior of $\bar{S}_b$. The result is a Fredholm first-kind integral equation for the unknown source strength, $\sigma(\vec{x},t) \equiv [\phi_n(\vec{x},t) - \phi'_n(\vec{x},t)]$:

$$\int\!\!\int_{S_b} d\vec{\xi}\,\sigma(\vec{\xi},t)\,G^{(0)}(\vec{x},\vec{\xi}) + \int_0^t d\tau \int\!\!\int_{S_b} d\vec{\xi}\,\sigma(\vec{\xi},\tau)\,G_t(\vec{x},\vec{\xi},t-\tau) = \phi(\vec{x},t).$$

[This can also be interpreted as an equation for the potential if the source strength is known.] Using the argument outlined above, the time integral is again a known quantity and so belongs on the right-hand side

$$\int\!\!\int_{S_b} d\vec{\xi}\,\sigma(\vec{\xi},t)\,G^{(0)}(\vec{x},\vec{\xi}) = g(\vec{x},t). \tag{5}$$

Equation (5) is the difference of the integral equation representations of the interior and the exterior Dirichlet problems, with both $\phi'(\vec{x},t)$ and $\phi(\vec{x},t)$ anti-symmetric about $z = 0$. Both of these equations have unique solutions and therefore Equation (5) also has a unique solution. Operating on Equation (5) with $\vec{n}\cdot\nabla_{\vec{x}}$ produces a Fredholm second-kind equation for the source strength that still has a unique solution.

The proposition that the transient integral equations have unique solutions appears to contradict the well-established existence of non-unique solutions to the corresponding integral equations in the frequency domain; however, the two views can be reconciled.

3 The Time-Harmonic Problem

If the motion of the fluid is assumed to be time harmonic at frequency ω then in the limit as $t \to \infty$ the potential will become $\Re\{\tilde{\phi}(\vec{x},\omega)\, e^{i\omega t}\}$ where $\tilde{\phi}$ is the solution to:

$$\begin{aligned} \nabla^2 \tilde{\phi} &= 0 \quad \text{in } \mathcal{V} \\ -\omega^2 \tilde{\phi} + g\tilde{\phi}_z &= 0 \quad \text{on } z = 0 \\ \vec{n} \cdot \nabla \tilde{\phi} &= \tilde{V}_n \quad \text{on } \bar{S}_b. \end{aligned} \tag{6}$$

As long as the boundary-value problem of Equation (6) is supplemented by a radiation condition to ensure outgoing waves as $R \to \infty$, then this is also a well-posed problem with a unique solution, and $\tilde{\phi}$ and ϕ can be considered to be a Fourier transform pair

$$\tilde{\phi}(\vec{x},\omega) = \int_{-\infty}^{\infty} dt \;\, \phi(\vec{x},t)\, e^{-i\omega t}, \qquad \phi(\vec{x},t) = \Re \int_{-\infty}^{\infty} d\omega \;\, \tilde{\phi}(\vec{x},\omega)\, e^{i\omega t}. \tag{7}$$

The appropriate Green function for the frequency-domain problem is

$$\tilde{G}(\vec{x},\vec{\xi},K) = \frac{1}{r} + \frac{1}{r'} + \frac{2K}{\pi} \int_0^{\infty} dk \;\, \frac{e^{kZ}}{k-K} J_0(kR), \tag{8}$$

where the contour of integration must be indented above the pole in order to satisfy the radiation condition, and $gK = \omega^2$. [The other quantities are identical to those defined in Equation (2).]

An integral equation representation of the boundary-value problem in frequency space may be derived as a distribution of wave sources and normal dipoles:

$$2\pi\tilde{\phi}(\vec{x},\omega) + \iint_{S_b} d\vec{\xi}\; \tilde{\phi}(\vec{\xi},\omega)\, \tilde{G}_{n_\xi}(\vec{x},\vec{\xi},\omega) = \iint_{S_b} d\vec{\xi}\; \tilde{\phi}_n(\vec{\xi},\omega)\, \tilde{G}(\vec{x},\vec{\xi},\omega), \tag{9}$$

or as source-only formulations:

$$\iint_{S_b} d\vec{\xi}\; \tilde{\sigma}(\vec{\xi},\omega)\, \tilde{G}(\vec{x},\vec{\xi},\omega) = \tilde{\phi}(\vec{x},\omega), \tag{10}$$

$$2\pi\tilde{\sigma}(\vec{x},\omega) + \iint_{S_b} d\vec{\xi}\; \tilde{\sigma}(\vec{\xi},\omega)\, \tilde{G}_{n_x}(\vec{x},\vec{\xi},\omega) = \vec{n} \cdot \nabla\tilde{\phi}(\vec{x},\omega). \tag{11}$$

It can be shown in several ways that these frequency-domain integral equations do not have unique solutions at an infinite discrete set of frequencies (the irregular frequencies) [1][4]. Away from the these frequencies the solutions are unique and coincide with the quantity of interest ($\tilde{\phi}$, or $\tilde{\sigma}$). *At* the irregular frequencies however, the potential formulation has an infinity of solutions, while the source-only equation has no solution at all (in general). *In the vicinity* of an irregular frequency, the source strength defined by Equation (11) is Cauchy-like with respect to the wavenumber (*i.e.* $\propto 1/(\omega_i^2 - \omega^2)$ where ω_i is the i^{th} irregular frequency), but the potential defined by either Equation (9) or the combination of (11) and (10) is well behaved [5]. It is clear that the quantities ϕ and $\tilde{\phi}$, (or σ and $\tilde{\sigma}$) as defined by boundary-value problems, are a Fourier transform pair; however, this property is not preserved when the same quantities are defined as solutions to integral equations.

4 Numerical Solutions

Consider the potential formulation in the frequency domain, Equation (9). The continuous solution to this equation is well behaved except at the irregular frequencies, while discrete numerical solutions show large errors in a finite bandwidth surrounding each of these frequencies. The bandwidth of this contaminated region can be made arbitrarily narrow by refining the spatial discretization, but the solution cannot be obtained at an irregular frequency. Calculations in the time domain are qualitatively similar, but fundamentally different. In transient numerical calculations, an oscillation can be observed (a superposition of irregular-frequency responses) which does not decay in time but whose amplitude can be reduced by refining the spatial and temporal discretizations. Thus discrete solutions in either the time or the frequency domain contain *numerical* irregular-frequency effects, and these effects are a Fourier transform pair. In the limit of infinitely fine discretization, the numerical irregular-frequency behavior disappears from the time-domain solution while it tends to the appropriate limit in the frequency domain.

We believe that the numerical irregular-frequency behavior of the discrete transient solution stems from an imperfect satisfaction of the initial conditions. This theory can be motivated by the following argument: It is well known that the irregular-frequency effects are related to the eigensolutions of an interior problem satisfying $\tilde{\phi}' = 0$ on $\bar{S}_b$, along with the linearized free-surface condition. These eigensolutions indicate that homogeneous solutions to the integral equation exist, and those solutions can be combined with the particular solution to produce an infinity of solutions at the irregular frequencies. If it were possible to construct a transient solution out of these eigensolutions, that satisfies the fictitious interior initial-boundary-value problem, then a non-trivial homogeneous solution would exist to the transient integral equation. However it is not possible to satisfy the exact initial conditions with such eigensolutions, and so a homogeneous solution to the transient integral equation can only be trivial. When the problem is solved numerically, there will be spatial and temporal discretization error with no particular frequency content, *and* the discrete enforcement of the initial conditions will admit a non-trivial homogeneous solution with its energy concentrated at the irregular frequencies. The former error will decay with increasing time as does the solution, while the latter error may not, as suggested by its Fourier transform pair.

References

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DISCUSSION

Yue D.K.P.: The frequency- and time-domain approaches to this problem are indentical. Irregular frequencies exist for both and are the same for either problem for the same geometry. The concerns are numerical in both cases in the sense that an analytic solution would be affected only at *discrete* (zero support) values of the frequency. The fact that the time-domain integral equation does not have unbounded solutions is however not sufficient for irregular frequencies to be absent. To show the latter, one must obtain that a hermonic respose of the time-domain integral equation (which acts like a feed-back loop) at any frequency decays sufficiently rapidly with time for the Fourier transform to exist. My question is how this is related to the initial conditon question you discussed.

Bingham H.: I would maintain that irregular frequencies do not exist in the analytic transient solution, however, you have a point. In addition to what has been shown here, we need to show that a solution like $ce^{i\omega_i t}$ can not exist in the transient solution. [For the potential formulation, the frequency-domain behavior is singular like $\frac{1}{\omega^2-\omega_i^2}$ where ω_i is an irregular frequency and this corresponds to a constant amplitude oscillation in the time-domain.] I should think more about this, but I believe this is ruled out by the initial conditions.

Yeung R.W.: Your conclusion suggests that oscillatory tail is associated with an imperfect initial condition at $t = 0^+$. What seems unclear is why imperfections in the evaluation of the time integral in each time step do not play a role at all. In my computations of the zero forward speed case (IUTAM Symp. on Wave Energy Utilization, 1985), I noticed the gradual growth and steadying of the oscillatory tail. What incited the growth is still not clear.

Bingham H.: I did not mean to suggest that only errors at the initial time step are involved. My view of it is that at any time t_1, the initial conditions have been satisfied only within some error which depends on the accumulated errors in the solution from $t = 0$ to $t = t_1$.